1) $\dot{x} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$

$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

a) $Ab = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$\Rightarrow C = (b \;\; Ab) = \boxed{\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}}$

b) $\operatorname{rank} C = \operatorname{rank} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = 1 < 2 = n$

So, since $\operatorname{rank} C < n$, the system is <u>not</u> controllable

c) not controllable so cannot drive $x_0 = \begin{pmatrix} 17 \\ 16.2 \end{pmatrix}$ to $x = 0$ in finite time. Also note that a controllable subspace is $S = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ and $x_0 = \begin{pmatrix} 17 \\ 16.2 \end{pmatrix} \notin S$.

2) $\ddot{q} = u \Rightarrow \begin{matrix} x_1 = q \\ x_2 = \dot{q} \end{matrix} \Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{b} u$

$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

So,

$$x(t) = \cancel{e^{At} x_0}^{\,0} + \int_0^t e^{A(t-\tau)} B u(t)\, d\tau$$

$$\therefore x(t) = \int_0^t e^{A(t-\tau)} B u(t)\, d\tau \qquad -(*)$$

$$C = (b \ \ Ab) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\Rightarrow$ rank $C = 2 = n$

So system <u>is</u> controllable / reachable

To find $u(t)$, let $u(t) = K =$ constant, and suppose $x(t)$ reaches $x_f = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ at time $t = T$.

Then, from $(*)$,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \int_0^T e^{A(T-\tau)} B\, u(\tau)\, d\tau$$

$$= \int_0^T \begin{pmatrix} 1 & T-\tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} K\, d\tau$$

So, $\begin{pmatrix}1\\1\end{pmatrix} = K\int_0^T \begin{pmatrix}T-\tau\\1\end{pmatrix} d\tau$

Thus, we have two equations:

$$1 = K\int_0^T (T-\tau)\,d\tau$$

$$1 = K\int_0^T d\tau$$

So, $1 = K\left(T\tau - \frac{\tau^2}{2}\Big|_0^T\right) = K\left(\frac{T^2}{2}\right)$

and $1 = KT$

Equating these two we have,

$$\cancel{K}T = \frac{1}{2}\cancel{K}T^2$$

$$\Rightarrow 1 = \frac{1}{2}T$$

$$\Rightarrow \underline{\underline{T=2}}$$

Since $KT=1$, then $K=\frac{1}{2}$. Thus, we can apply a constant control $u=K=\frac{1}{2}$ for $0\le t\le 2$ to reach $x_f=\begin{pmatrix}1\\1\end{pmatrix}$.

So, $\boxed{u(t) = \begin{cases}\frac{1}{2}, & 0\le t\le 2\\ 0, & t>2\end{cases} \text{ will work}}$

3) $\dot{x} = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix} x$

a) $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\det(\lambda I - A_1) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1 = 0 \Rightarrow \boxed{\lambda_\pm = \pm 1}$

$A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\det(\lambda I - A_2) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \boxed{\lambda_\pm = \pm i}$

b) $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\chi_{A_1}(\lambda) = \lambda^2 - \cancel{\operatorname{tr} A_1}^{\,0}\lambda + \det A_1 = \boxed{\lambda^2 - 1}$

$A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\chi_{A_2}(\lambda) = \lambda^2 - \cancel{\operatorname{tr} A_2}^{\,0} + \det A_2 = \boxed{\lambda^2 + 1}$

c) comparing (b) + (c), we can see that

for $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\det(\lambda I - A_1) = \lambda^2 - 1 = \chi_{A_1}(\lambda)$

and for $A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\det(\lambda I - A_2) = \lambda^2 + 1 = \chi_{A_2}(\lambda)$

```
%ESE 441 Homework 4, Problem 4 Solution

%A matrix
A = [0 1 1 0;
     0 0 1 0;
     0 0 0 1;
     1 2 3 4];

%b matrix
b = [0;
     0;
     0;
     1];

%controllability matrix
C = [b A*b A*A*b A*A*A*b];

%rank of controllability matrix
rank_C = rank(C);

%since rank(C) = 4 = n, the system is completely controllable


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%place closed-loop poles at p = (-1 -1 -1.5 -2)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%use Ackermann's formula:

%desired closed loop eigenvalues
p =[-1; -1; -1.5; -2];

%compute feedback gain
K = acker(A,b,p)

%closed loop eigenvalues
eig_CL = eig(A-b*K)



C =

     0     0     1     5
     0     0     1     4
     0     1     4    19
     1     4    19    91


rank_C =

     4


K =

    4.0000    8.5000   14.0000    9.5000


eig_CL =

   -2.0000
```

-1.5000
-1.0000
-1.0000