1) i) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$sI - A = sI = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \Rightarrow (sI-A)^{-1} = \begin{pmatrix} 1/s & 0 \\ 0 & 1/s \end{pmatrix}$$

$$\boxed{\therefore e^{At} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Can also notice that $e^{At} = e^{0} \equiv I$

<u>or</u> $e^{At} \equiv I + \cancel{At}^{0} + \frac{A^2 t^2}{2} + \cdots = I$

ii) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$sI - A = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} s & -1 \\ 0 & s \end{pmatrix}$$

$$(sI-A)^{-1} = \frac{1}{s^2}\begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{pmatrix}$$

$$\Rightarrow e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \boxed{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}$$

Notice that A is nilpotent, i.e. $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$

So,

$$e^{At} = I + At + \cancel{\frac{A^2 t^2}{2}}^{0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}$$

iii) $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$

$\therefore \sigma(A) = \{-1, -2\}$.

Thus $e^{At} = P e^{\Lambda t} P^{-1}$, where $e^{\Lambda t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$

and P is the matrix of eigenvectors of A.

$\underline{\lambda_1 = -1}$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow v_2 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

So take $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\underline{\lambda_2 = -2}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow v_1 = 0 \Rightarrow \vec{v}_2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$

So take $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Let $P = [\vec{v}_1 \; \vec{v}_2] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow P^{-1} = I$.

$$\therefore e^{At} = e^{\Lambda t} = \boxed{\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}}$$

Note: This can be computed directly from $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$
since A is diagonal!

i.e.

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{\begin{pmatrix} (-1)^k & 0 \\ 0 & (-2)^k \end{pmatrix} t^k}{k!}$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(-2)^k t^k}{k!} \end{pmatrix} \equiv \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

iv). $A = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$

$$sI - A = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} s+1 & -1 \\ 0 & s+2 \end{pmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{pmatrix} s+2 & 1 \\ 0 & s+1 \end{pmatrix} = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{pmatrix}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}, \quad A = \frac{\cancel{(s+1)}}{\cancel{(s+1)}(s+2)}\Big|_{s=-1} = 1$$

$$B = \frac{(s+2)}{(s+1)\cancel{(s+2)}}\Big|_{s=-2} = -1$$

So, $(sI - A)^{-1} = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+2} \\ 0 & \frac{1}{s+2} \end{pmatrix}$

$$\therefore e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \boxed{\begin{pmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{pmatrix}}$$

v) $A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$.

$$sI - A = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} s+1 & 0 \\ -1 & s+2 \end{pmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{pmatrix} s+2 & 0 \\ 1 & s+1 \end{pmatrix} = \begin{pmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+2} \end{pmatrix}.$$

$$= \begin{pmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s+1} - \frac{1}{s+2} & \frac{1}{s+2} \end{pmatrix}.$$

$$\therefore\ e^{At} = \mathcal{L}^{-1}\left\{ (sI - A)^{-1} \right\} = \boxed{\begin{pmatrix} e^{-t} & 0 \\ e^{-t} - e^{-2t} & e^{-2t} \end{pmatrix}}$$

2) $\dot{x} = \underbrace{\begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}}_{A} x, \quad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$sI - A = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} s+1 & -2 \\ 2 & s+1 \end{pmatrix}$$

$$(sI-A)^{-1} = \frac{1}{(s+1)^2+2^2}\begin{pmatrix} s+1 & 2 \\ -2 & s+1 \end{pmatrix} = \begin{pmatrix} \frac{s+1}{(s+1)^2+2^2} & \frac{2}{(s+1)^2+2^2} \\ \frac{-2}{(s+1)^2+2^2} & \frac{s+1}{(s+1)^2+2^2} \end{pmatrix}$$

$$e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \begin{pmatrix} e^{-t}\cos 2t & e^{-t}\sin 2t \\ -e^{-t}\sin 2t & e^{-t}\cos 2t \end{pmatrix}$$

$$\Rightarrow x(t) = e^{At}x_0 = e^{At}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} e^{-t}\sin 2t \\ e^{-t}\cos 2t \end{pmatrix}}$$

$$\chi_A(s) = \lambda^2 - \operatorname{tr}A\,\lambda + \det A = \lambda^2 + 2\lambda + 5 = 0$$

$$\therefore \lambda_{\pm} = \frac{-2 \pm \sqrt{4-4(5)}}{2} = \boxed{-1 \pm 2i}$$

Since $Re(\lambda_{\pm}) < 0$, the system is <u>asympt. stable</u>

3) $\ddot{y} + 2\dot{y} + 5y = 0$.

Take Laplace Transform:

$\Rightarrow \chi(s) = s^2 + 2s + 5 = 0$.

Same as in Problem 2:

$$\boxed{\lambda_\pm = -1 \pm 2i}$$

System is asympt. stable since $Re(\lambda_\pm) < 0$.

<u>Note</u>: since $a_1 = 2 > 0$ and $a_2 = 5 > 0$, the system is asympt. stable by Routh-Hurwitz

3) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$e^{At} = e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix}$$

$$\Rightarrow x(t) = e^{At} x_0 = e^{At} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}}$$

Although $x(t) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} \to 0$ as $t \to \infty$, the system is not asympt. stable since one of the eigenvalues of $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ has positive real part. Actually, the x_1-axis is a stable axis and the x_2-axis is an unstable axis.

Extra Credit Problem

a) $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$.

$$sI\ A = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = \begin{pmatrix} s & -a \\ a & s \end{pmatrix}$$

$$(sI-A)^{-1} = \frac{1}{s^2+a^2}\begin{pmatrix} s & a \\ -a & s \end{pmatrix} = \begin{pmatrix} \frac{s}{s^2+a^2} & \frac{a}{s^2+a^2} \\ \frac{-a}{s^2+a^2} & \frac{s}{s^2+a^2} \end{pmatrix}$$

$$e^{At} = \mathcal{L}^{-1}\{(sI-A)\} = \boxed{\begin{pmatrix} \cos at & \sin at \\ -\sin at & \cos at \end{pmatrix}}$$

b) $e^{Bt} = \boxed{\begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}}$.

c) $A+B = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = \begin{pmatrix} 0 & a+b \\ -(a+b) & 0 \end{pmatrix}$

d) replacing a with a+b in part (a) or (b) results in

$$\boxed{e^{A+B} = \begin{pmatrix} \cos((a+b)t & \sin((a+b)t) \\ -\sin((a+b)t) & \cos((a+b)t) \end{pmatrix}}$$

e) Show $e^{A+B} = e^A e^B \Rightarrow AB = BA$.

Pf. To show $AB = BA$ we first need a Lemma:

Lemma: $\frac{d}{dt} e^{At} = A e^{At}$.

Pf. $\frac{d}{dt} e^{At} = \lim_{h \to 0} \frac{e^{A(t+h)} - e^{At}}{h} = \lim_{h \to 0} \frac{(e^{Ah} - I)}{h} e^{At}$

$= \lim_{h \to 0} \left(A + \frac{A^2 h}{2!} + \frac{A^3 h^2}{3!} + \cdots \right) e^{At}$

$= A e^{At}$

Note: $\frac{d}{dt} e^{At} = e^{At} A$ also.

Suppose $e^{(A+B)t} = e^{At} e^{Bt}$. — $(*)$

Now differentiate both sides of $(*)$ to obtain

$(A+B)(A+B) e^{(A+B)t} = A^2 e^{At} e^{Bt} + A e^{At} B e^{Bt} + A e^{At} B e^{Bt}$
$+ e^{At} B^2 e^{Bt}$

— $(**)$

Since $(**)$ holds for all $t \in \mathbb{R}$, we can evaluate $(**)$ at $t=0$ to obtain,

$$(A+B)(A+B) = A^2 + AB + AB + B^2$$

$$\Rightarrow A^2 + AB + BA + B^2 = A^2 + AB + AB + B^2.$$

Subtract $A^2 + AB + B^2$ from both sides and we obtain

$$\underline{\underline{AB = BA}}$$

e) $e^A e^B = \begin{pmatrix} \cos at & \sin at \\ -\sin at & \cos at \end{pmatrix} \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}$

$= \begin{pmatrix} \cos at \cos bt - \sin at \sin bt & \cos at \sin bt + \sin at \cos bt \\ -\sin at \cos bt - \cos at \sin bt & -\sin at \sin bt + \cos at \cos bt \end{pmatrix}$

$= \begin{pmatrix} \cos(a+b)t & \sin(a+b)t \\ -\sin(a+b)t & \cos(a+b)t \end{pmatrix}$

$= e^{A+B}$

$\begin{cases} \sin(a+b) = \sin a \cos b + \cos a \sin b \\ \cos(a+b) = \cos a \cos b - \sin a \sin b. \end{cases}$