FATOU'S LEMMA FOR UNBOUNDED GELFAND INTEGRABLE MAPPINGS

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ABSTRACT. It is shown that, in the framework of Gelfand integrable mappings, the Fatou-type lemma for integrably bounded mappings, due to Cornet and Medecin [?] and the Fatou-type lemma for norm uniformly integrable mappings due to Balder [?], can be generalized to mean norm bounded integrable mappings.

1. Introduction

In General Equilibrium Theory, the theoric framework to model perfect competition is to consider a measure space of agents. For economies with finitely many commodities, the Fatou type Lemma of Arstein [?] has been of most importance to prove the existence of equilibria. Gelfand integration appeared to be the appropiate concept in many models with infinitely many commodities (see for instance Ostroy and Zame [?], Podczeck [?] and Cornet and Médecin [?]).

Podczeck [?] and Cornet and Medecin [?] proved a Fatou-type lemma for integrably bounded mappings. Balder [?] generalized these results for norm uniformly integrable mappings. In mathematical economic models, *extra* assumptions (boundedness of consumption sets, strict monotonicity of preferences) where introduced in order to apply these theorems.

We propose in this paper to generalize the Fatou-type Lemma of Cornet and Medecin [?] and Balder [?] to mean norm bounded integrable mappings. Moreover we provide a simple condition under wich mean norm boundedness of a sequence of mappings is implied by the boundedness of the sequence of means.

This result should enable us to substantially weaken the monotonicity assumptions used in Mas-Colell [?], Jones [?], Ostroy and Zame [?], Podczeck [?], Cornet and Médecin [?] and Martins Da Rocha [?].

2. Statement of results

2.1. **Gelfand integrable mappings.** In the whole paper we assume that $(\Omega, \mathcal{A}, \mu)$ is a finite complete positive measure space, $(E, \|.\|)$ is a separable Banach space, with topological dual space E^*. We shall mainly consider on the space E^* the weak star topology $\sigma(E^*, E)$, noted w^*, and we shall use the notation $\lim$, cl (etc..) to specify the limit, the closure of a set (etc..) for this topology. For $x \in E, f \in E^*$, we denote by $\langle x, f\rangle := f(x)$ the dual product, and by $\|.\|^*$ the dual norm on E^*, i.e., $\|f\|^* := \sup_{x\neq 0} |\langle f, x\rangle|/\|x\|$. We denote by B and B^*, the closed unit balls in $(E, \|.\|)$ and $(E^*, \|.\|^*)$, respectively. If (x_k) is a sequence in E^* we denote by $\mathrm{Ls}_k\{x_k\}$ the set of w^*-limit points of $\{x_k\}$. If D is a subset of E^* and $x \in E$, we let $\delta^*(x, D) := \sup\{\langle x, x'\rangle \mid x' \in D\}$. If $C \subset E$ (resp. $C^* \subset E^*$) is a subset of E, then we note $C^\circ \subset E^*$ (resp. $[C^*]^\circ$) the negative polar cone of C (resp. C^*), that is, $x^* \in C^0$ (resp. $x \in [C^*]^\circ$) iff for all $x \in C$ (resp. $x^* \in C^*$), $\langle x, x^*\rangle \leqslant 0$. If $F \subset E$ is a subspace of E, then the negative polar F° coincide with the orthogonal $F^\perp$ defined by $\{x^* \in E^* \mid \langle x, x^*\rangle = 0 \quad \forall x \in F\}$. Note that if A is a subset of E^*, then

$$A \subset \bigcap_{F \in \mathcal{F}} [A + F^{\perp}] \subset \operatorname{cl} A,$$

where $\mathcal{F}$ is the collection of all finite dimensional subspaces of E. In particular if E is finite dimensional, then $A = \bigcap_{F \in \mathcal{F}} [A + F^{\perp}]$.

A mapping f from Ω to E^* is said to be Gelfand measurable[1], if for every $x \in E$, the real valued function $a \mapsto \langle x, f(a) \rangle$ is measurable, and f is said to be Gelfand integrable, if for every $x \in E$, the function $a \mapsto \langle x, f(a) \rangle$ is integrable. If f is Gelfand integrable, it can be shown (see Diestel and Uhl [?] pp. 52-53) that for each $A \in \mathcal{A}$, there exists a unique $x_A^* \in E^*$ such that

$$\forall x \in E \quad \langle x, x_A^* \rangle = \int_{\Omega} \langle x, f(a) \rangle d\mu(a).$$

For each $A \in \mathcal{A}$, x_A^* is noted $\int_A f d\mu$. Note that if f is a Gelfand measurable mapping, then the function $a \mapsto \|f(a)\|^*$ is mesurable [2]. However if f is Gelfand integrable then $a \mapsto \|f_n(a)\|^*$ is not necessary integrable. A Gelfand measurable mapping f is said norm integrable if $a \mapsto \|f(a)\|^*$ is integrable. Obviously, a norm integrable mapping is Gelfand integrable.

A sequence (f_n) of Gelfand integrable mappings from Ω to E^* is said mean norm bounded, if

$$\sup_n \int_{\Omega} \|f_n(a)\|^* \, d\mu(a) < +\infty.$$

Let C be cone (of vertex 0) of E. A sequence (f_n) of Gelfand integrable mappings from Ω to E^* is said C-uniformly integrable if for all $x \in E$, the sequence

$$(\langle x, f_n(a) \rangle^-)_n \quad \text{is uniformly integrable,}$$

where $\langle x, f_n(a) \rangle^- := \max[0, -\langle x, f_n(a) \rangle]$.

2.2. **Fatou's lemma.**

Theorem 2.1 (Convex Fatou's Lemma)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space and $(E, \|.\|)$ be a separable Banach space. Let (f_n) be a sequence of Gelfand integrable mappings from Ω to E^*, which is mean norm bounded and C-uniformly integrable for a cone $C \subset E$.*

If $\lim_n \int_{\Omega} f_n d\mu$ exists in E^* then there exists a Gelfand integrable mapping f such that

$$\int_{\Omega} f d\mu - \lim_n \int_{\Omega} f_n d\mu \in C^0$$

and

$$f(a) \in \overline{\operatorname{co}} \operatorname{Ls}_n \{f_n(a)\} \quad \text{a.e.}$$

In fact f is norm integrable and

$$\int_{\Omega} \|f(a)\|^* \, d\mu(a) \leqslant \sup_n \int_{\Omega} \|f_n(a)\|^* \, d\mu(a).$$

Theorem 2.1 is a direct consequence of Theorem 3.1 in Section 3.

Theorem 2.2 (Approximate Fatou's Lemma)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space, let $(E, \|.\|)$ be a separable Banach space, and let (f_n) be a sequence of Gelfand integrable mappings from Ω to E^*, which is mean norm bounded and C-uniformly integrable for a convex cone $C \subset E$.*

[1] We prove in Appendix that f is Gelfand measurable if and only if for each borelian $B \subset E^*$, $f^{-1}(B) \in \mathcal{A}$.

[2] See Proposition A.1 in Appendix.

If $\lim_n \int_\Omega f_n d\mu$ *exists in* E^* *then for each finite dimensional subspace* F *of* E*, there exists a Gelfand integrable mapping* f_F *such that*

$$\int_\Omega f_F d\mu - \lim_n \int_\Omega f_n d\mu \in C^\circ + F^\perp$$

and

$$f_F(a) \in \mathrm{Ls}_n\{f_n(a)\} \quad \text{a.e.}$$

In particular, if the dimension of E *is finite, then there exists a Gelfand integrable mapping* f_E *such that*

$$\int_\Omega f_E d\mu - \lim_n \int_\Omega f_n d\mu \in C^\circ$$

and

$$f_E(a) \in \mathrm{Ls}_n\{f_n(a)\} \quad \text{a.e.}$$

In fact for each finite dimensional subspace F*, the mapping* f_F *is norm integrable and*

$$\int_\Omega \|f_F(a)\|^* \, d\mu(a) \leqslant \sup_n \int_\Omega \|f_n(a)\|^* \, d\mu(a).$$

Theorem 2.2 will be proved in Section 5 as a Corollary of Theorem 3.1.

2.3. **The case of C^*-uniformly integrable mappings.** We present hereafter a corollary of Theorems 2.1 and 2.2 which gives a sufficient condition for a sequence of mappings to satisfy the assumptions of Theorem 2.1 and Theorem 2.2.

Corollary 2.1. *Let* (f_n) *be a sequence of Gelfand integrable mappings from* Ω *to* E^* *such that*

$$\forall n \in \mathbb{N} \quad f_n(a) \in C^* + \varphi_n(a) B^* \quad \text{a.e.},$$

where C^* *is a locally* w^**-compact*[3] *pointed*[4] *closed convex cone in* E^* *and* (φ_n) *is a sequence of uniformly integrable positive functions. Suppose that* $\lim_n \int_\Omega f_n d\mu$ *exists in* E^**.*

(1) [Convex Fatou's Lemma]. *There exists a Gelfand integrable mapping* f *such that*

$$\int_\Omega f d\mu - \lim_n \int_\Omega f_n d\mu \in -C^*$$

and

$$f(a) \in \overline{\mathrm{co}}\, \mathrm{Ls}_n\{f_n(a)\} \quad \text{a.e.}$$

(2) [Approximate Fatou's Lemma]. *For every finite dimensional subspace* F *of* E*, there exists a Gelfand integrable mapping* f_F *such that*

$$\int_\Omega f_F d\mu - \lim_n \int_\Omega f_n d\mu \in F^\perp - C^*$$

and

$$f_F(a) \in \mathrm{Ls}_n\{f_n(a)\} \quad \text{a.e.}$$

(3) [Finite Fatou's Lemma]. *If* E *is finite dimensional then there exists a Gelfand integrable mapping* f_* *such that*

$$\int_\Omega f_* d\mu - \lim_n \int_\Omega f_n d\mu \in -C^*$$

and

$$f_*(a) \in \mathrm{Ls}_n\{f_n(a)\} \quad \text{a.e.}$$

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[3]A subset C^* of E^* is said locally w^*-compact if each point of C^* has a compact neighborhood for the induced w^*-topology on C^*.

[4]A closed convex cone $C^* \subset E^*$ is said to be pointed if it contains no line, that is, if $C^* \cap (-C^*) = \{0\}$.

Proof. Let C^* be a pointed closed convex cone which is locally w^*-compact. Applying the following Proposition 2.1, there exists $e \in E$ such that

$$\forall x^* \in C^* \setminus \{0\} \quad \langle e, x^* \rangle > 0$$

and the following set S, defined by

$$S := \{x^* \in C^* \mid \langle e, x^* \rangle = 1\}$$

is w^*-compact. It follows that S is $\|.\|^*$-bounded by $m > 0$. In particular,

$$\langle e, x^* \rangle \geqslant m \|x^*\|^* \quad \forall x^* \in C^*.$$

For each $n \in \mathbb{N}$, we consider the following correspondance $F_n : a \mapsto C^* \cap [\{f_n(a)\} - \varphi_n(a)B^*]$. Applying Theorem A.1, there exists $c_n : \Omega \mapsto C^*$ and $b_n : \Omega \mapsto B^*$, two measurable mappings such that for all $n \in \mathbb{N}$,

$$\forall a \in \Omega \quad f_n(a) = c_n(a) + \varphi_n(a) b_n(a).$$

Since the sequence $(\int_\Omega f_n d\mu)$ converges, we can then suppose (passing to a subsequence if necessary) that the sequences $(\int_\Omega c_n d\mu)$ and $(\int_\Omega \varphi_n b_n d\mu)$ converges in E^*. Now, let $v^* := \lim_n \int_\Omega c_n d\mu$, then

$$\lim_n \int_\Omega \|c_n(a)\|^* d\mu(a) \leqslant \frac{1}{m} \langle e, v^* \rangle$$

and the sequence (c_n) is mean norm bounded. It follows that the sequence (f_n) is mean norm bounded. Now let $C \subset E$ be the negative polar of C^*. Then C is a convex cone of vertex 0 and (f_n) is $[-C]$-uniformly integrable. It follows that the conditions of Theorems 2.1 and 2.2 are met. Noticing that the negative polar of $(-C)$ is $-C^*$, then the corollary follows. □

Remark 2.1. If E is finite dimensional, then every pointed closed convex cone is locally compact. In particular, Corollary 2.1 generalizes the version of Fatou's Lemma proved in Cornet and Topuzu [**?**].

Remark 2.2. Let T be a compact metric space and let $E = C(T)$ be the separable Banach space of continuous real-valued functions endowed with the supremum norm. The topological dual space E^* is then $M(T)$, the space of finite Radon measures on T. Let $C := C(T)_+$ and $C^* = M(T)_+$ be the natural positive cones of $C(T)$ and $M(T)$, respectively, that is, $C(T)_+ := \{x \in C(T) \mid x(t) \geqslant 0 \text{ for every } t \in T\}$, and $M(T)_+ := \{f \in M(T) \mid \langle x, f \rangle \geqslant 0 \text{ for every } x \in C(T)_+\}$. Then $M(T)_+$ is a locally w^*-compact pointed closed convex cone. In particular, Corollary 2.1 can be applied in mathematical economics to prove the existence of Walras equilibria for large square economies with differentiated commodities (see Martins Da Rocha [**?**]).

Proposition 2.1. *Let L be a Hausdorff locally convex topological vector space. Let C be a locally compact closed convex cone (of vertex 0) in L which is pointed, i.e., $C \cap -C = \{0\}$. Then there exists a continuous linear form $f \in L'$ such that*

$$\forall x \in C \setminus \{0\} \quad f(x) > 0.$$

Moreover the set $S := \{x \in C \mid f(x) = 1\}$ is compact.

Proof. Since C is locally compact, there exists W a neighborhood of 0 in L such that $W \cap C$ is compact. The topological vector space L is locally convex, hence there exists an open convex neighborhood V of 0 in L such that $V \subset W$. It follows that A, the closure of $V \cap C$, is compact convex. Let $x \in A \setminus \{0\}$, since L is Hausdorff, there exists a continuous linear form $f_x \in L'$ such that $f_x(x) > 0$. It follows that

$$A \setminus [(1/2)V)] \subset A \setminus \{0\} \subset \bigcup_{f \in L'} A \cap \{f > 1\}.$$

Since A is compact, it follows that there exists a finite $\{f_1, \ldots, f_n\} \subset L'$ such that

$$A \setminus [(1/2)V)] \subset \bigcup_{i=1}^{n} A \cap \{f_i > 1\}.$$

We let B be convex hull of $\bigcup_{i=1}^{n} A \cap \{f_i \geqslant 1\}$. The set B is compact convex and since C is pointed, $0 \notin B$. Applying Hahn Banach separation theorem, there exists a continuous linear form $f \in L'$ such that

$$\forall x \in B \quad f(x) > 0.$$

In particular, for all $x \in C \setminus \{0\}$, $f(x) > 0$. It follows that

$$A \setminus [(1/2)V)] \subset \bigcup_{\alpha > 0} C \cap \{f > \alpha\}.$$

Since A is compact, there exists a finite set $\{\alpha_1, \ldots, \alpha_n\}$ such that

$$A \setminus [(1/2)V)] \subset \bigcup_{i=1}^{n} C \cap \{f > \alpha_i\}.$$

In particular if we let $\alpha := \min_i \alpha_i$, then

$$A \setminus [(1/2)V)] \subset C \cap \{f > \alpha\}.$$

Since A is convex, it follows that

$$C \cap \{f = \alpha/2\} \subset A \quad \text{is compact.}$$

□

Remark 2.3. Proposition 2.1 was proposed as an exercice in Bourbaki [**?**].

2.4. **The link with other results.** We recall the following notions about sequences of integrable mappings.

Definition 2.1. *A sequence (f_n) of Gelfand integrable mappings from Ω to E^* is said*

(1) integrably bounded *if there exists an integrable function $\varphi \in \mathcal{L}^1_{\mathbb{R}}(\Omega, \mathcal{A}, \mu)$ such that*

$$\sup_n \|f_n(a)\|^* \leqslant \varphi(a) \quad \text{a.e. },$$

(2) uniformly integrable *if*

$$\lim_{\alpha \to \infty} \sup_n \int_{\{\|f_n\|^* \geqslant \alpha\}} \|f_n(a)\|^* \, d\mu(a) = 0 \ ,$$

(3) mean norm bounded *if*

$$\sup_n \int_\Omega \|f_n(a)\|^* \, d\mu(a) < +\infty \ ,$$

Remark 2.4. Following Neveu [**?**], we recall that the sequence (φ_n) of real valued functions is uniformly integrable if and only if the sequence is mean norm bounded, i.e.,

$$\sup_n \int_\Omega |\varphi_n| d\mu < +\infty$$

and *equi-continuous*, i.e., for every $\varepsilon > 0$ there exists $\eta > 0$ such that for all $A \in \mathcal{A}$,

$$\mu(A) \leqslant \eta \Longrightarrow \sup_n \int_A |\varphi_n| d\mu \leqslant \varepsilon.$$

It follows that if (f_n) is a sequence of Gelfand integrable mappings from Ω to E^*, then the integrably boundedness condition is stronger than the uniform integrability condition which is stronger than the mean norm boundedness condition.

Definition 2.2. *A sequence (f_n) of Gelfand integrable mappings from Ω to E^* is said*

(1) norm uniforlmy integrable *if the sequence $(\|f_n(.)\|^*)$ is uniformly integrable, i.e.*
$$\lim_{\alpha\to+\infty}\int_{\{\|f_n\|^*>\alpha\}} \|f_n(a)\|^*\, d\mu(a) = 0.$$
(2) C^*-uniformly integrable *for a cone $C^* \subset E^*$, if there exists an uniformly integrable sequence of positive functions (φ_n) such that*
$$f_n(a) \subset C^* + \varphi_n(a)B^* \quad \text{a.e.}$$
(3) C-uniformly integrable *for a cone $C \subset E^*$, if for all $x \in C$, the sequence $(\langle x, f_n(.)\rangle^-)$ is uniformly integrable, where*
$$\forall a \in \Omega \quad \langle x, f_n(a)\rangle^- := \max[0, -\langle x, f_n(a)\rangle].$$

Remark 2.5. Let (f_n) be a sequence of Gelfand integrable mappings from Ω to E^*. If (f_n) is norm uniformly integrable, then (f_n) is $\{0_{E^*}\}$-uniformly integrable. If (f_n) is C^*-uniformly integrable with $C^* \subset E^*$ a cone of E^*, then (f_n) is C-uniformly integrable, where C is the negative polar of $-C^*$. In particular, if (f_n) is norm uniformly integrable, then (f_n) is E-uniformly integrable.

Remark 2.6. In the context of Balder [**?**], the sequence (f_n) is supposed to be uniformly integrable. It follows that the sequence (f_n) is mean norm bounded and E-uniformly integrable. Hence Theorems 2.1 and 2.2 generalize Theorems 1 and 2 in [**?**]. Remarquons que contrairement á Balder, on a mieux que $f_W(a) \in \mathrm{cl}\,\mathrm{Ls}_n\{f_n(a)\}$

Remark 2.7. In the context of Cornet and Médecin [**?**], the sequence (f_n) is supposed to be integrably bounded. As previously, it follows that the sequence (f_n) is mean norm bounded and E-uniformly integrable. Hence Theorems 2.1 and 2.2 generalize Theorem 1 in [**?**].

2.5. **The case of $E = \mathbb{R}$.** We provide hereafter the generalized Fatou's lemma and the Fatou's lemma for real valued functions.

Corollary 2.2. *Let (φ_n) be a sequence of integrable functions from Ω to $\mathbb{R}$.*

(1) *Suppose that (φ_n^-) is uniformly integrable, where*
$$\forall n \in \mathbb{N} \quad \forall a \in \Omega \quad \varphi_n^-(a) := \max[0, -\varphi_n(a)]\ ;$$
and that
$$\liminf_n \int_\Omega \varphi_n d\mu > -\infty.$$
Then there exits an integrable function φ_* such that
$$\varphi_*(a) \in \mathrm{Ls}_n\{\varphi_n(a)\} \quad \text{a.e.}$$
and
$$\int_\Omega \varphi_* d\mu \leqslant \liminf_n \int_\Omega \varphi_n d\mu.$$
(2) *Suppose that the sequence (φ_n^-) is integrably bounded, i.e. there exists an integrable function ρ such that*
$$\forall n \in \mathbb{N} \quad \forall a \in \Omega \quad \varphi_n(a) \geqslant \rho(a).$$
Then there exits an integrable function φ_* such that
$$\varphi_*(a) \in \mathrm{Ls}_n\{\varphi_n(a)\} \quad \text{a.e.}$$
and
$$\int_\Omega \varphi_* d\mu \leqslant \liminf_n \int_\Omega \varphi_n d\mu.$$

In both cases (1) and (2), we get

$$\int_\Omega \liminf_n \varphi_n d\mu \leqslant \liminf_n \int_\Omega \varphi_n d\mu.$$

Proof. In both cases (1) and (2), we have that

$$\liminf_n \int_\Omega \varphi_n d\mu > -\infty$$

in particular, there exists a subsequence (k) of (n) such

$$\liminf_n \int_\Omega \varphi_n d\mu = \lim_k \int_\Omega \varphi_k d\mu.$$

Since the sequence (φ_k^-) is mean norm bounded, it follows that the sequence (φ_k) is mean norm bounded. Applying Theorem 2.2 to the sequence (φ_k), we get the case (1). Applying Corollary 2.1 to the sequence (φ_k), we get the case (2). □

3. Generalization of Komlós' Theorem

To prove Fatou's Lemma we propose to provide an extension to vector-valued mappings, of the important result by Komlós (Theorem A.2 in Appendix), proved for real-valued functions. We first recall the following definition of Komlós convergence or simply K−convergence.

Definition 3.1. *A sequence (f_m) of mappings from Ω to E^* is said to be* K-*convergent almost everywhere to a mapping $f : \Omega \to E^*$, denoted*

$$f_m \xrightarrow{\mathrm{K}} f,$$

if for every subsequence (m_i) of (m), there exists a null set $N \in \mathcal{A}$ (i.e., $\mu(N) = 0$) such that

$$\forall a \in \Omega \setminus N, \ \ (1/n)\sum_{i=1}^{n} f_{m_i}(a) \xrightarrow{w^*} f(a).$$

Theorem 3.1 (Komlós' Theorem)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space, let $(E, \|.\|)$ be a separable Banach space, and let (f_n) be a sequence of Gelfand integrable mappings from Ω to E^*, which is mean norm bounded. Then there exists a subsequence (m) of (n) and a Gelfand integrable mapping f such that (f_m)* K*-converge to f, i.e., for all (m_i) subsequence of (m),*

$$\frac{1}{n}\sum_{i=1}^{n} f_{m_i}(a) \xrightarrow{w^*} f(a) \quad \text{a.e.}$$

Moreover, there exists ρ a positive integrable function such that for all finite dimensional subspace F of E,

$$f(a) \in \mathrm{coLs}_n\{f_n(a)\} + \rho(a)B^* \cap F^\perp \quad \text{a.e.},$$

in particular

$$f(a) \in \overline{\mathrm{co}}\,\mathrm{Ls}_n\{f_n(a)\} \quad \text{a.e.}$$

In fact f is norm integrable and

$$\int_\Omega \|f(a)\|^* \, d\mu(a) \leqslant \sup_n \int_\Omega \|f_n(a)\|^* \, d\mu(a).$$

If moreover the sequence (f_n) is C-uniformly integrable for a cone $C \subset E$, then

$$\forall A \in \mathcal{A} \quad \forall x \in C \qquad \int_A \langle x, f(a)\rangle d\mu(a) \leqslant \liminf_n \int_A \langle x, f_m(a)\rangle d\mu(a).$$

The proof of Theorem 3.1 is given in Section 4.

Remark 3.1. Theorem 3.1 provides an extension to vector-valued mappings, of the important result by Komlós (Theorem A.2 in Appendix), proved for real-valued functions. We also refer to Balder [**?**] and Balder and Hess [**?**] for other extensions of Komlós' result, which mainly consider Bochner integration instead of Gelfand's as below.

Remark 3.2. Note that Theorem 2.1 is a direct consequence of Theorem 3.1.

We recall that a sequence (f_m) of Gelfand integrable mappings from Ω to E^* is said to be weakly convergent to a Gelfand integrable mapping f, if (f_m) is mean norm bounded and if

$$\forall A \in \mathcal{A} \quad \lim_m \int_A f_m d\mu = \int_A f d\mu.$$

Remark 3.3. A sequence (f_m) of Gelfand integrable mappings from Ω to E^* is weakly convergent to a Gelfand integrable mapping f, if and only for all $x \in E$, the sequence of real valued functions $a \mapsto \langle x, f_n(a)\rangle$ converges to the function $a \mapsto \langle x, f(a)\rangle$ for the weak topology $\sigma(L^1_{\mathbb{R}}, L^\infty_{\mathbb{R}})$ (cf Theorem 7 p. 1291 in Dunford and Schwartz [**?**]).

A direct consequence of Theorem 2.1 is the following weak sequential compactness criterium.

Corollary 3.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space and $(E, \|.\|)$ be a separable Banach space. If $\mathcal{H}$ is a family of Gelfand integrable mappings from Ω to E^* which are mean norm bounded and E-uniformly integrable, then $\mathcal{H}$ is weak sequentially compact.*

Proof. Indeed, if (f_n) is a mean norm bounded sequence of Gelfand integrable mappings, then applying Theorem 2.1, there exists a Gelfand integrable mapping f and a subsequence (m) of (n) such that (f_m) K-converges to f. Moreover if (f_n) is E-uniformly integrable, then

$$\forall A \in \mathcal{A} \quad \int_A f d\mu = \lim_m \int_A f_m d\mu.$$

In particular (f_m) weakly converges to f. □

Remark 3.4. A sequence (f_n) of Gelfand integrable mappings is said norm uniformly integrable if the sequence $(\|f_n(.)\|^*)$ is uniformly integrable, i.e.,

$$\lim_{\alpha \to +\infty} \int_{\|f_n(.)\|^* > \alpha} \|f_n(a)\|^* \, d\mu(a) = 0.$$

Following Neveu [**?**], we recall that the sequence (φ_n) of real valued functions is uniformly integrable if and only if the sequence is mean norm bounded, i.e.,

$$\sup_n \int_\Omega |\varphi_n| d\mu < +\infty$$

and *equi-continuous*, i.e., for every $\varepsilon > 0$ there exists $\eta > 0$ such that for all $A \in \mathcal{A}$,

$$\mu(A) \leqslant \eta \Longrightarrow \sup_n \int_A |\varphi_n| d\mu \leqslant \varepsilon.$$

It follows that if the sequence (f_n) of Gelfand integrable mappings is norm uniformly integrable, then (f_n) is mean norm bounded and E-uniformly integrable. In particular, if $\mathcal{H}$ is a family of norm uniformly integrable mappings, then $\mathcal{H}$ is sequentially weakly compact.

4. Proof of Theorem 3.1

We prepare the proof of Theorem 3.1 by several results.

Proposition 4.1. *Let (f_k) be a sequence of Gelfand integrable mappings from Ω to E^*, which is mean norm bounded. Then there exists a subsequence (m) of (k) and a Gelfand integrable function $f : \Omega \to E^*$ such that the sequence (f_m)* K*-converges to f. Moreover the mapping f is in fact norm integrable and*

$$\int_\Omega \|f(a)\|^* \, d\mu(a) \leqslant \sup_n \int_\Omega \|f_n(a)\|^* \, d\mu(a).$$

Proof. We let for each $k \in \mathbb{N}$,

$$\forall a \in \Omega \quad \psi_k(a) := \|f_k(a)\|^* .$$

Let (x_j) be a $\|.\|$-dense sequence in E. We define for each $j, k \in \mathbb{N}$

$$\varphi_{j,k}(a) := \langle x_j, f_k(a)\rangle \quad \text{and} \quad \varphi_{\infty,k} := \psi_k$$

Since the sequence (f_k) is mean norm bounded, then for every $j \in \mathbb{N} \cup \{\infty\}$,

$$\sup_k \int_\Omega \mid \varphi_{j,k}(a) \mid d\mu(a) < +\infty.$$

It is now possible to apply Komlós' Theorem (Theorem A.2 in Appendix) repeatedly in a diagonal procedure. This yields a subsequence (m) of (k) and a family $(\varphi_j)_{j\in\mathbb{N}\cup\{\infty\}}$ of integrable real valued functions such that for every $j \in \mathbb{N} \cup \{\infty\}$ and every subsequence (m_i) of (m)

$$\frac{1}{n}\sum_{i=1}^{n} \varphi_{j,m_i}(a) \longrightarrow \varphi_j(a) \quad \text{a.e.}$$

In particular, for every $j \in \mathbb{N}$

$$\langle x_j, \frac{1}{n}\sum_{i=1}^{n} f_{m_i}(a)\rangle \longrightarrow \varphi_j(a) \quad \text{a.e.} \tag{1}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} \psi_{m_i}(a) \longrightarrow \varphi_\infty(a) \quad \text{a.e.} \tag{2}$$

Fix $a \in \Omega$ outside the exceptional null-set and for each $n \in \mathbb{N}$, define

$$g_n(a) := \frac{1}{n}\sum_{m=1}^{n} f_m(a).$$

Then, applying (2), $\limsup_n \|g_n(a)\|^* \leqslant \varphi_\infty(a) < +\infty$. Now applying Banach-Alaoglu's Theorem, there exists a subsequence of $(g_n(a))$ converging for the w^*-topology to some $f(a) \in E^*$. Applying (1), for every $j \in \mathbb{N}$

$$\langle x_j, f(a)\rangle = \varphi_j(a).$$

The sequence (x_j) is $\|.\|$-dense in E, it follows that for every subsequence (m_i) of (m)

$$\frac{1}{n}\sum_{i=1}^{n} f_{m_i}(a) \xrightarrow{w^*} f(a) \quad \text{a.e.} \tag{3}$$

In particular, the mapping f is Gelfand mesurable. Now $\|f(a)\|^* \leqslant \liminf_n \|g_n(a)\|^*$ almost every where in Ω. Hence applying Fatou's Lemma for positive functions,

$$\int_\Omega \|f(a)\|^* \, d\mu(a) \leqslant \liminf_n \int_\Omega \|f_n(a)\|^* \, d\mu(a) < +\infty$$

and the mapping f is then Gelfand integrable. □

Proposition 4.2. *Let $\{f_k\}$ be a sequence of Gelfand integrable mappings from Ω to E^*, K-converging to a Gelfand integrable mapping f. Suppose that there exists a cone $C \subset E$ such that (f_k) is C-uniformly integrable. Then for all $x \in C$,*

$$\forall A \in \mathcal{A} \quad \int_A \langle x, f(a)\rangle d\mu(a) \leqslant \liminf_k \int_A \langle x, f_k(a)\rangle d\mu(a).$$

Proof. Let $A \in \mathcal{A}$ and $x \in C$, we pose $\alpha := \liminf_k \int_A \langle x, f_k\rangle d\mu$. There exists a subsequence (ℓ) of (k) such that

$$\alpha = \lim_\ell \int_A \langle x, f_\ell(a)\rangle d\mu(a).$$

We define, for each k, the fonction φ_k from Ω to $\mathbb{R}$, by:

$$\varphi_k(a) := -\langle x, f_k(a)\rangle^- = -\max[0, -\langle x, f_k(a)\rangle].$$

Note that the sequence of real valued functions (φ_k) is uniformly integrable over A and for every k, $\langle x, f_k(a)\rangle \geqslant \varphi_k(a)$ for each $a \in A$. Applying Komlós' Theorem (Theorem A.2 in Appendix), there exists a subsequence (m) of (ℓ) and an integrable real valued function φ such that

$$\frac{1}{n}\sum_{m=1}^{n} \varphi_m(a) \longrightarrow \varphi(a) \quad \text{for a.e. } a \in A.$$

Hence, by uniform integrability, it follows that

$$\frac{1}{n}\sum_{m=1}^{n} \int_A \varphi_m d\mu \longrightarrow \int_A \varphi d\mu,$$

so it follows that

$$\alpha - \int_A \varphi d\mu = \lim_n \left[\frac{1}{n}\sum_{m=1}^{n} \int_A (\langle x, f_m(a)\rangle - \varphi_m(a)) d\mu(a)\right].$$

Since the sequence (f_k) is K-convergent to f,

$$\frac{1}{n}\sum_{m=1}^{n} \langle x, f_m(a)\rangle \longrightarrow \langle x, f(a)\rangle \quad \text{for a.e. } a \in A.$$

Since for each m, $\langle x, f_m(a)\rangle \geqslant \varphi_m(a)$, we apply Fatou's Lemma (for real valued functions)

$$\alpha - \int_A \varphi d\mu \geqslant \int_A [\langle x, f\rangle - \varphi]\, d\mu = \int_A \langle x, f\rangle d\mu - \int_A \varphi d\mu.$$

Consequently

$$\alpha := \liminf_k \int_A \langle x, f_k(a)\rangle d\mu(a) \geq \int_A \langle x, f(a)\rangle d\mu(a).$$

□

Proposition 4.3. *Let E be a finite dimensional vector space and let (f_n) be a sequence of integrable mappings from Ω to E^*. Suppose that the sequence (f_n) is mean norm bounded and is K-converging to an integrable mapping f. Then*

$$f(a) \in \mathrm{co}\mathrm{Ls}_n\{f_n(a)\} \quad \text{a.e.}$$

Proof. Let (f_n) be a sequence of mean norm bounded mappings from Ω to E^*, K-converging to 0.

Step 1: the sequence (f_n) is norm uniformly integrable. Suppose in the contrary that there exists $A \in \mathcal{A}$, with $\mu(A) \neq 0$ and such that

$$\forall a \in A \quad 0 \notin \mathrm{coLs}_n\{f_n(a)\}.$$

Without any loss of generality, we can suppose that $A = \Omega$. Applying a separation argument, for each $a \in \Omega$, there exists $x(a) \in E$, with $\|x(a)\| = 1$ and such that

$$0 \geqslant \delta^* \left(x(a), \mathrm{Ls}_n\{f_n(a)\}\right) := \sup\{\langle x(a), x^*\rangle \mid x^* \in \mathrm{Ls}_n\{f_n(a)\}\}. \tag{4}$$

Applying Proposition A.4 and Theorem A.1, we can choose the mapping $x : \Omega \to E$ to be measurable.

Claim 4.1. *There exists a subsequence (m) of (n) such that*

$$\mathrm{Ls}_m\{f_m(a)\} \subset \{x(a)\}^\perp \quad \text{a.e..}$$

Proof. For each $n \in \mathbb{N}$ and each $\alpha > 0$, let $A_{n,\alpha} := \{a \in \Omega \mid \|f_n(a)\|^* \geqslant \alpha\}$. The set $A_{n,\alpha}$ is measurable and

$$\alpha\mu(A_{n,\alpha}) \leqslant \sup_n \int_\Omega \|f_n(a)\|^* \, d\mu(a).$$

It follows that $\lim_{\alpha\to\infty} \mu(A_{n,\alpha}) = 0$ uniformly in $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ and each $\beta > 0$, let $B_{n,\beta} := \{a \in \Omega \mid \langle x(a), f_n(a)\rangle \geqslant \beta\}$. The set $B_{n,\beta}$ is measurable. We propose to prove that $\lim_n \mu(B_{n,\beta}) = 0$, for each $\beta > 0$. Indeed, let $\varepsilon > 0$, there exists $\alpha > 0$ such that $\mu(B_{n,\beta} \cap A_{n,\alpha}) \leqslant \varepsilon/2$. Now let $B'_{n,\beta} := B_{n,\beta} \setminus A_{n,\alpha}$. Suppose that the sequence $(\mu(B'_{n,\beta}))_n$ does not converge to 0, then passing to a subsequence if necessary, we can suppose that there exists $\delta > 0$ such that for all $n \in \mathbb{N}$, $\mu(B'_{n,\beta}) \geqslant \delta$. In particular

$$\mu(\limsup_n B'_{n,\beta}) \geqslant \limsup_n \mu(B'_{n,\beta}) \geqslant \delta,$$

where $\limsup_n B'_{n,\beta} := \bigcap_{n\in\mathbb{N}} \bigcup_{k\geqslant n} B'_{k,\beta}$. Note that

$$\limsup_n B'_{n,\beta} \subset \{a \in \Omega \mid \exists x^* \in \mathrm{Ls}_n\{f_n(a)\},\ \langle x(a), x^*\rangle \geqslant \beta > 0\}.$$

This contradicts (4). Hence there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geqslant n(\varepsilon)$, $\mu(B_{n,\beta}) \leqslant \varepsilon/2 + \varepsilon/2$.

For each $n \in \mathbb{N}$ and each $\gamma > 0$, let $C_{n,\gamma} := \{a \in \Omega \mid \langle x(a), f_n(a)\rangle \leqslant -\gamma\}$. The set $C_{n,\gamma}$ is measurable. We propose to prove that $\lim_n \mu(C_{n,\gamma}) = 0$ for each $\gamma > 0$. Suppose that the sequence $(\mu(C_{n,\gamma}))_n$ does not converge to 0, then passing to a subsequence if necessary, we can suppose that there exists $\delta > 0$ such that for all $n \in \mathbb{N}$, $\mu(C_{n,\gamma}) \geqslant \delta$. We can suppose that $\mu(\Omega) = 1$. It follows then

$$\int_{C_{n,\gamma}} \langle x(a), f_n(a)\rangle d\mu(a) \leqslant -\delta\gamma < 0. \tag{5}$$

Now let $\beta > 0$, we propose to prove that for all n large enough,

$$\int_{\Omega\setminus C_{n,\gamma}} \langle x(a), f_n(a)\rangle d\mu(a) \leqslant 2\beta. \tag{6}$$

Indeed, since the sequence (f_n) is norm uniformly integrable, then it is equi-continuous. Since we proved that $\lim_n \mu(B_{n,\beta}) = 0$, then for n large enough

$$\int_{B_{n,\beta}} \|f_n(a)\|^* \, d\mu(a) \leqslant \beta.$$

Moreover

$$\int_{\Omega\setminus(B_{n,\beta}\cup C_{n,\gamma})}\langle x(a), f_{n(a)}\rangle d\mu(a) \leqslant \beta.$$

It follows that (6) is satisfied. Now choosing β small enough, we proved by (5) and (6) that for all n large enough,

$$\int_{\Omega}\langle x(a), f_n(a)\rangle d\mu(a) \leqslant -\delta\gamma/2 < 0. \tag{7}$$

But the sequence (f_n) is uniformly integrable and K-converges to 0. Applying Proposition 4.2, the sequence (f_n) weakly converges to 0, in particular

$$\lim_{n\to\infty}\int_{\Omega}\langle x(a), f_n(a)\rangle d\mu(a) = 0.$$

This contradicts (7).

We have proved that for each $\beta > 0$ and $\gamma > 0$,

$$\lim_{n\to\infty}\mu(B_{n,\beta}) = 0 \quad \text{and} \quad \lim_{n\to\infty}\mu(C_{n,\gamma}) = 0.$$

This means that the sequence of functions $(\langle x(.), f_n(.)\rangle)$ converges in measure to 0. Following Proposition II.4.3 in Neveu [**?**], there exists a subsequence (m) of (n), such that

$$\lim_{m\to\infty}\langle x(a), f_m(a)\rangle = 0 \quad \text{a.e.}$$

It follows in particular that for almost every $a \in \Omega$, if $x^* \in \mathrm{Ls}_m\{f_m(a)\}$ then $\langle x(a), x^*\rangle = 0$. □

We now come back to the proof of Step 1. Since (m) is a subsequence of (n), the sequence (f_m) is mean norm bounded and K-converging to 0. Moreover for all $a \in \Omega$, $0 \notin \mathrm{Ls}_m\{f_m(a)\}$. However, the dimension of $\{x(a)\}^{\perp}$ is strictly less than the dimension of E^*, and following a classical induction argument, we get a contradiction.

Step 2: The general case. Following Gaposhkin's Lemma A.1, there exists a subsequence (n_k) of (n) such that for each $k \in \mathbb{N}$, $f_{n_k} = g_k + h_k$, where the sequence (g_k) is uniformly integrable and the sequence (h_k) converges almost every where to 0. Since (f_{n_k}) K-converges to 0, it follows that (h_k) K-converges to 0. Now applying Step 1, $0 \in \mathrm{coLs}_k\{g_k(a)\}$ almost every where. Since $\mathrm{Ls}_k\{g_k(a)\} \subset \mathrm{Ls}_n\{f_n(a)\}$ almost every where, it follows that $0 \in \mathrm{coLs}_n\{f_n(a)\}$ almost every where. □

Remark 4.1. Page in [**?**] (Proposition 1) proposed a proof of Proposition 4.3 based on the finite dimensional Fatou's Lemma of Artstein [**?**]. Note that Propositions 4.1, 4.2 and 4.3 give an alternate proof of the finite dimensional Fatou's Lemma proved by Artstein [**?**] and Balder [**?**].

Remark 4.2. Note Proposition C in Artstein [**?**] is a corollary of Propositions 4.1 4.2 and 4.3. Indeed, let (f_n) be sequence of integrable mappings from Ω to E^* (E is finite dimensional), such that (f_n) weakly converges to an integrable mapping f. The sequence (f_n) is then mean norm bounded, applying Propositions 4.1 and 4.3, there exists a subsequence (m) of (n) and an integrable mapping g such that (f_m) K converge to g and that $g(a) \in \mathrm{coLs}_m\{f_m(a)\}$ almost every where. But since (f_n) weakly converges, it follows from Proposition IV.2.3 in Neveu [**?**] that (f_n) is uniforlmy integrable. Applying Proposition 4.2, the sequence (f_m) weakly converges to g. Hence $g = f$ almost every where and $f(a) \in \mathrm{coLs}_n\{f_n(a)\}$ almost every where.

Proposition 4.4. *Let (f_n) be a sequence of Gelfand integrable mappings from Ω to E^*. Suppose that the sequence (f_n) is mean norm bounded and is* K*-converging to a Gelfand integrable mapping f. Then there exists ρ a positive integrable function such that for all finite dimensional subspace F of E,*

$$f(a) \in \mathrm{coLs}_n\{f_n(a)\} + \rho(a)B^* \cap F^\perp \quad \text{a.e.}$$

Proof. Let F be a finite dimensional subspace of E. We consider π the following projection from E^* to F^*, defined by

$$\forall x^* \in E^* \quad \pi(x^*) = [x \in F \mapsto \langle x, x^* \rangle].$$

The sequence $(\|f_n(.)\|^*)$ is mean norm bounded, following Proposition 4.1, passing to a subsequence if necessary, we can suppose that the sequence

$$\left[\|f_n(.)\|^*, \pi(f_n)\right]$$

is mean norm bounded and is K-converging to $[\psi, \pi(f)]$, where ψ is an integrable function from Ω to $\mathbb{R}$. Applying Proposition 4.3,

$$[\psi(a), \pi(f(a))] \in \mathrm{coLs}_n\left\{\left[\|f_n(a)\|^*, \pi(f_n(a))\right]\right\} \quad \text{a.e.}$$

Let $a \in \Omega$ outside the exceptional null set. There exists a finite set I, a finite family $(\lambda_i)_{i\in I} \in [0,1]^I$ such that $\sum_{i\in I} \lambda_i = 1$, and there exists a finite family $(\varphi_i)_{i\in I}$ of strictly increasing functions from $\mathbb{N}$ to $\mathbb{N}$, such that

$$[\psi(a), \pi(f(a))] = \sum_{i\in I} \lambda_i \lim_n \left[\left\|f_{\varphi_i(n)}(a)\right\|^*, \pi(f_{\varphi_{i(n)}}(a))\right].$$

Let $i \in I$, the sequence $(\left\|f_{\varphi_i(n)}(a)\right\|^*)$ converges, passing to a subsequence if necessary, we can suppose that the sequence $(f_{\varphi_i(n)}(a))$ w^*-converges to $h_i(a) \in \mathrm{Ls}_n\{f_n(a)\} \subset E^*$. It follows that

$$\pi[f(a)] = \sum_{i\in I} \lambda_i \pi[h_i(a)] \in \pi\left[\mathrm{coLs}_n\{f_n(a)\}\right].$$

Note that $\left\|\sum_{i\in I} \lambda_i h_i(a)\right\|^* \leqslant \psi(a)$, hence

$$f(a) \in \mathrm{coLs}_n\{f_n(a)\} + \rho(a)B^* \cap F^\perp,$$

where $\rho(a) := \psi(a) + \|f(a)\|^*$. □

Proposition 4.5. *Let L be a multifunction from Ω to E^*, let f be a mapping from Ω to E^* and let ρ a positive function such that for every finite dimensional subspace F of E,*

$$f(a) \in L(a) + \rho(a)B^* \cap F^\perp \quad \text{a.e.}$$

Then

$$f(a) \in \mathrm{cl}\, L(a) \quad \text{a.e.}$$

Proof. Let (e_i) be a dense sequence in E, and for each $n \in \mathbb{N}$, we let F_n be the vector subspace of E generated by $\{e_0, e_1, \dots, e_n\}$. It follows that there exists $\Omega' \subset \Omega$ with $\mu(\Omega \setminus \Omega') = 0$ and such that

$$\forall a \in \Omega' \quad f(a) \in \bigcap_{n\in\mathbb{N}} \left(L(a) + \rho(a)B^* \cap F_n^\perp\right).$$

Let $a \in \Omega'$ and let W be a w^*-neighborhood of zero in E^*. There exists a finite set $\{x_1, \dots, x_\ell\} \subset E$ such that

$$\{x^* \in E^* \mid |\langle x_i, x^* \rangle| \leqslant 1 \quad i = 1, \dots, \ell\} \subset W.$$

For each $i \in \{1, \dots, \ell\}$ there exists $n_i \in \mathbb{N}$ such that $\|x_i - e_{n_i}\| < 1/\rho(a)$ (without any loss of generality, we can suppose that $\rho(a) > 0$). Let $n_0 := \max\{n_i \mid i =$

$1,\dots,\ell\}$, there exists $g(a) \in L(a)$ and $h(a) \in \rho(a)B^* \cap F_{n_0}^{\perp}$ such that $f(a) = g(a) + h(a)$. For each $i \in \{1,\dots,\ell\}$,

$$|\langle x_i, h(a)\rangle| \leqslant |\langle e_{n_i}, h(a)\rangle| + \|h(a)\|^* \|x_i - e_{n_i}\| < 1.$$

Hence $h(a) \in W$ and $f(a) \in L(a) + W$. It follows that

$$\forall a \in \Omega' \quad f(a) \in \bigcap_{W \in \mathcal{V}} L(a) + W = \operatorname{cl} L(a),$$

where $\mathcal{V}$ is the collection of all w^*-neighborhoods of 0 in E^*. □

Now the proof of Theorem 3.1 follows directly Propositions 4.1, 4.2, 4.4 and 4.5.

5. Proof of Theorem 2.2

5.1. **The case $(\Omega, \mathcal{A}, \mu)$ is non atomic.** Let (f_n) be a sequence of Gelfand integrable mappings, which is mean norm bounded, and let F be a finite dimensional subspace of E. Applying Theorem 2.1, we can suppose, passing to a subsequence if necessary that there exists f a Gelfand integrable mapping from Ω to E^* and ψ an integrable function from Ω to $[0, +\infty[$ such that

$$(\|f_n(.)\|^*, f_n) \xrightarrow{\mathrm{K}} (\psi, f) \quad \text{a.e.}$$

and

$$(\psi(a), f(a)) \in \operatorname{coLs}_n\{(\|f_n(a)\|^*, f_n(a))\} + (\mathbb{R} \times F)^{\perp} \quad \text{a.e.}$$

Let π be the following projection from E^* to F^*, the algebraic dual of F, defined by

$$\forall x^* \in E^* \quad \pi(x^*) = [x \in F \mapsto \langle x, x^*\rangle].$$

Then

$$(\psi(a), \pi[f(a)]) \in \operatorname{coLs}_n\{(\|f_n(a)\|^*, \pi[f_n(a)])\} \quad \text{a.e.}$$

Following Carathéodory's theorem, we let $I := \{1, \dots, \ell+2\}$, where ℓ is the dimension of F. Then, for almost every $a \in \Omega$, there exists $(\lambda_i(a))_{i\in I} \in [0,1]^I$ such that $\sum_{i\in I} \lambda_i(a) = 1$ and $(\varphi_i)_{i\in I}$ strictly increasing functions from $\mathbb{N}$ to $\mathbb{N}$, such that

$$(\psi(a), \pi[f(a)]) = \sum_{i\in I} \lambda_i(a) \lim_n (\left\|f_{\varphi_i(n)}(a)\right\|^*, \pi[f_{\varphi_i(n)}(a)]).$$

In particular, for each $i \in I$, $\lim_n \left\|f_{\varphi_i(n)}(a)\right\|^* < +\infty$, it follows that there exists $s_i(a) \in \operatorname{Ls}_n\{f_n(a)\}$ such that $\lim_n f_{\varphi_i(n)}(a) = s_i(a)$, and

$$(\psi(a), \pi[f(a)]) = \sum_{i\in I} \lambda_i(a)(\|s_i(a)\|^*, \pi[s_i(a)]) \quad \text{a.e.}$$

Applying Proposition A.4 and Corollary A.1, we can suppose that for each $i \in I$, the functions λ_i are measurable and the mappings s_i are Gelfand measurable selections of $\operatorname{Ls}_n\{f_n(.)\}$. Note that for each $i \in I$, for almost every $a \in \Omega$, $\|s_i(a)\|^* \leqslant \liminf_n \left\|f_{\varphi_i(n)}(a)\right\|^*$. It follows that

$$\int_\Omega \sum_{i\in I} \lambda_i(a) \|s_i(a)\|^* \leqslant \int_\Omega \psi(a) d\mu(a) < +\infty$$

and hence that

$$\int_\Omega \sum_{i\in I} \lambda_i(a) \left|(\|s_i(a)\|^*, \pi[s_i(a)])\right| d\mu(a) \leqslant 2 \int_\Omega \psi d\mu < \infty.$$

Applying the Extended Lyapunov Theorem A.3, there exists a measurable partition $(B_i)_{i\in I}$ of Ω such that $(\|s_i(.)\|^*, \pi[s_i(.)])$ is integrable over B_i and such that

$$\int_\Omega \sum_{i\in I} \lambda_i(a)(\|s_i(a)\|^*, \pi[s_i(a)]) d\mu(a) = \sum_{i\in I} \int_{B_i} (\|s_i(a)\|^*, \pi[s_i(a)]) d\mu(a).$$

Let $f_F := \sum_{i\in I} \chi_{B_i} s_i$ where χ_A is the characteristic function of the measurable set $A \in \mathcal{A}$. Then f_F is a Gelfand measurable selection of $\mathrm{Ls}_n\{f_n(.)\}$, and moreover

$$\int_\Omega \|f_F(a)\|^* \, d\mu(a) = \sum_{i\in I} \int_{B_i} \|s_i(a)\|^* \, d\mu(a) \leqslant \int_\Omega \psi d\mu < \infty.$$

It follows that f_F is Gelfand integrable. Now

$$\pi[\int_\Omega f_F d\mu] = \sum_{i\in I} \int_{B_i} \pi[s_i(a)] d\mu(a) = \int_\Omega \sum_i \lambda_i(a) \pi[s_i(a)] d\mu(a) = \int_\Omega \pi[f(a)] d\mu(a).$$

Hence

$$\int_\Omega f d\mu - \int f_F d\mu \in F^\perp.$$

5.2. **The general case.** We now give to the proof of Theorem 2.2 in the general case, i.e., without assuming anymore that $(\Omega, \mathcal{A}, \mu)$ is non-atomic. This is a classical result that the set Ω can be partitioned into a non atomic part $\Omega^{na} \in \mathcal{A}$ and a purely atomic part $\Omega^{pa} \in \mathcal{A}$, and that the set Ω^{pa} can be written as the disjoint union of at most countably many measurable atoms $(A_i)_{i\in I}$ $(I \subset N)$. Furthermore, for every $i \in I$ and every $n \in N$, the measurable mapping $f_n : \Omega \to E^*$ takes a constant value $f_n^i \in E^*$ for a.e. $a \in A_i$. Since the sequence (f_n) is mean norm bounded, for each $i \in I$, the sequence (f_n^i) is norm bounded, and thus remains in a w^*-compact subset of E^* by Alaoglu's theorem. Consequently, by a diagonal extraction argument, there exists a subsequence (n_k) of (n) such that for every $i \in I$, $(f_{n_k}^i)_k$ w^*-converges to some element $\bar{f}^i \in E^*$. We let $f^{pa} : \Omega^{pa} \to E^*$ be defined by $f^{pa}(a) = \bar{f}^i$ if $a \in A^i$, and we have shown that

$$f^{pa}(a) \in \mathrm{Ls}_n\{f_n(a)\} \quad \text{a.e. in } \Omega^{pa}.$$

We now show that

$$\lim_k \int_{\Omega^{pa}} f_{n_k}(a) d\mu(a) = \int_{\Omega^{pa}} f^{pa}(a) d\mu(a).$$

This is clearly a consequence of Lebesgue dominated convergence theorem applied for every fixed $x \in E$, to the sequence $(\langle x, f_{n_k}\rangle)_k$ which is integrably bounded over Ω^{pa} (since the sequence $(f_{n_k})_k$ is also integrably bounded over Ω^{pa}).

We now consider the non atomic part Ω^{na} and we first remark that $\lim_k \int_{\Omega^{na}} f_{n_k} d\mu$ exists since

$$\lim_k \int_{\Omega^{na}} f_{n_k} d\mu = \lim_k \int_\Omega f_{n_k} d\mu - \lim_k \int_{\Omega^{pa}} f_{n_k} d\mu.$$

We can now apply to the non atomic part Ω^{na} the version of Fatou's lemma proved previously. Thus, for every finite dimensional subspace F of E, there exists $f_F^{na} : \Omega^{na} \to E^*$ such that

$$f_F^{na}(a) \in \mathrm{Ls}_n\{f_n(a)\} \quad \text{a.e. in } \Omega^{na}$$

and

$$\int_{\Omega^{na}} f_F^{na} d\mu - \lim_k \int_{\Omega^{na}} f_{n_k} d\mu \in F^\perp + C^\circ.$$

We now define the mapping $f_F : \Omega \to E^*$ by $f_F(a) := f^{pa}(a)$ if $a \in \Omega^{pa}$ and $f_F(a) := f_F^{na}(a)$ if $a \in \Omega^{na}$. One checks that the mapping f_F satisfies the condition of Theorem 2.2.

Appendix A. Appendix

A.1. **Measurable mappings.** Let $(\Omega, \mathcal{A}, \mu)$ be a complete finite measure space and E a separable Banach space. We note $\mathcal{B}$ the Borel σ-algebra of E^*. We recall that a mapping f from Ω to E^* is said Gelfand measurable if for all $x \in E$, the function $a \mapsto \langle x, f(a) \rangle$ is measurable. The mapping f is said measurable, if for all $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$, where $\mathcal{B}$ is the Borel σ-algebra.

Proposition A.1. *Let f be a mapping from Ω to E^*. Then f is Gelfand measurable if and only if f is measurable. Moreover, if f is measurable, then the function $a \mapsto \|f(a)\|^*$ is measurable.*

Proof. Let (x_i) a norm dense sequence in B the unit ball of E. For each $i \in \mathbb{N}$ and each $\alpha > 0$, we let $V_{i,\alpha} := \{x^* \in E^* \mid |\langle x_i, x^* \rangle| < \alpha\}$. We note $\mathcal{D}$ the σ-algebra generated by the family of all $V_{i,\alpha}$. Since $V_{i,\alpha}$ is open in (E^*, w^*), we have $\mathcal{D} \subset \mathcal{B}$. It follows that if f is measurable then f is Gelfand measurable. Note that

$$\bigcup_{i \in \mathbb{N}} \bigcap_{n>0} V_{i,\alpha+1/n} = \alpha B^* = \{x^* \in E^* \mid \|x^*\|^* \leqslant \alpha\} \in \mathcal{D}.$$

It follows that if f is Gelfand measurable then the mapping $a \mapsto \|f(a)\|^*$ is measurable.

Let d be the following distance defined on E^*,

$$\forall (x^*, y^*) \in E^* \times E^* \quad d(x^*, y^*) = \sum_{i \geqslant 0} \frac{|\langle x_i, x^* - y^* \rangle|}{2^i}.$$

Let $\mathcal{B}_d$ be the Borel σ-algebra defined by d. Note that $\mathcal{B}_d \subset \mathcal{D}$. The topology defined by the distance d coincide with w^* topology on closed bounded subsets of E^*. It follows that if W is a w^* subset of E^*, then for each $k \in \mathbb{N}$, $W \cap kB^*$ is d-open, in particular, $W \cap kB^* \in \mathcal{D}$. Since $W = \bigcup_k W \cap kB^*$, it follows that $W \in \mathcal{D}$, and then $\mathcal{B} \subset \mathcal{D}$. Hence $\mathcal{B} = \mathcal{D}$ and the result follows. □

A.2. **Measurable selections.** Let $(\Omega, \mathcal{A}, \mu)$ be a complete finite measure space and E a separable Banach space. A multifunction F from Ω into E^* is said graph measurable if the graph Graph (F) of F belongs to the σ-algebra $\mathcal{A} \otimes \mathcal{B}$, where

$$G_F := \{(a, x^*) \in \Omega \times E^* \mid x^* \in F(a)\}.$$

A mapping f from Ω to E^* is a selection of F if $f(a) \in F(a)$ for almost every $a \in \Omega$. We provide hereafter a classical result of existence of measurable selections.

Theorem A.1 (Aumann Selection Theorem)**.** *We consider E a separable Banach space and $(\Omega, \mathcal{A}, \mu)$ a complete finite measure space. Let F be a graph measurable multifunction from Ω to E^* with non empty values. Then there exists a measurable mapping f from Ω to E^* such that*

$$\forall a \in \Omega \quad f(a) \in F(a),$$

in particular f is a measurable selection of F.

The proof of this theorem is given in Castaing and Valadier [**?**] (Theorem III.22 p. 74). We provide hereafter a direct application of this theorem.

Corollary A.1. *We consider E a separable Banach space and $(\Omega, \mathcal{A}, \mu)$ a complete finite measure space. Let F be a graph measurable multifunction from Ω to E^* with non empty values and let I be a finite set and let f be a measurable selection of F. Suppose that for almost every $a \in \Omega$, for each $i \in I$, there exist $\lambda_i(a) \in [0,1]$ and $f_i(a) \in F(a)$ such that*

$$f(a) = \sum_{i \in I} \lambda_i(a) f_i(a) \quad \textit{and} \quad \sum_{i \in I} \lambda_i(a) = 1.$$

Then for each $i \in I$, λ_i may be choosen as a measurable function from Ω to $[0,1]$ and f_i may be choosen as measurable selection of F.

Proof. We let $\Sigma(I)$ be the set of all $(\alpha_i) \in [0,1]^I$ such that $\sum_i \alpha_i = 1$. Let π be the linear mapping from $\Sigma(I) \times (E^*)^I$ to E^* defined by

$$\forall [(\alpha_i),(x_i^*)] \in \Sigma(I) \times (E^*)^I \quad \pi[(\alpha_i),(x_i^*)] := \sum_{i \in I} \alpha_i x_i^*.$$

For each $a \in \Omega$, we let

$$H(a) := \pi^{-1}(\{f(a)\}) \cap \left(\Sigma(I) \times F(a)^I\right).$$

The multifunction H is graph measurable with non empty values. The proof of the corollary follows from the application of Theorem A.1 to the multifunction H. □

A.3. **Mesurability of limes superior.** A multifunction F from Ω into E^* is said measurable if for each w^*-open subset V of E^*,

$$F^-(V) := \{a \in \Omega \mid F(a) \cap V \neq \emptyset\} \in \mathcal{A}.$$

Proposition A.2. *Let F be a multifunction from Ω to E^*.*

(1) *Suppose that $(\Omega, \mathcal{A}, \mu)$ is complete. If F is graph measurable then F is measurable.*
(2) *Suppose that F is closed valued. If F is measurable then F is graph measurable.*

Proof. The part (1) follows from the Projection Theorem (Theorem III.23) in Castaing and Valadier [**?**]. Since E is a separable Banach space, E^* is the countable union of w^*-compact metrisable subsets. It follows from Schwartz [**?**] that E^* is a Lusin space, in particular, there exists a separable and completely metrizable topology τ, stronger than the w^* topology, but generating the same Borel sets. Since F is w^*-closed valued, it is τ-closed valued. Applying Proposition III.13 in Castaing and Valadier [**?**], the graph of F is measurble. □

Proposition A.3. *Let F and F_n, $n \in \mathbb{N}$ be graph measurable multifunctions from Ω into E^*.*

(1) *The multifunction $\operatorname{cl} F$ defined by $a \mapsto \operatorname{cl} F(a)$ is graph measurable.*
(2) *The multifunction $\bigcup_n F_n$ and $\bigcap_n F_n$ are graph measurable.*

Proof. Proof of (1). The multifunction F is graph measurable, and then following Proposition A.2, F is measurable. Let V be a w^*-open subset of E^*. For each $a \in A$,

$$F(a) \cap V \neq \emptyset \Longleftrightarrow [\operatorname{cl} F(a)] \cap V \neq \emptyset.$$

It follows that if F is measurable then cl$F$ is measurable. Once again applying Proposition A.2, the multifunction cl$F$ is graph measurable.
Proof of (2). This an immediate consequence of

$$\operatorname{Graph}(\cup_n F_n) = \cup_n \operatorname{Graph}(F_n) \quad \text{and} \quad \operatorname{Graph}(\cap_n F_n) = \cap_n \operatorname{Graph}(F_n)$$

□

If (C_n) is a sequence of subsets of E^*, we note $\mathrm{Ls}_n C_n$ the sequential limes superior of (C_n) relative to w^*, i.e.,

$$\mathrm{Ls}_n C_n := \{x \in E^* \mid x = \lim_k x_k\,,\ x_k \in C_{n_k}\}.$$

Proposition A.4. *Let (F_n) be a sequence of graph measurable multifunctions from Ω into E^*. The multifunction $a \mapsto \mathrm{Ls}_n F_n(a)$ is graph measurable. In particular, if (f_n) is a sequence of measurable mappings from Ω to E^*, then the multifunction $a \mapsto \mathrm{Ls}_n\{f_n(a)\}$ is graph measurable.*

Proof. Note that if (C_n) is a sequence of non-empty subsets of E^*, then

$$\mathrm{Ls}_n C_n = \bigcup_{p\in\mathbb{N}} \mathrm{Ls}_n(C_n \cap pB^*).$$

Indeed, let $x \in \mathrm{Ls}_n C_n$. There exists a sequence (x_k) and a subsequence (n_k) of (n) such that $x_k \in C_{n_k}$ for each $k \in \mathbb{N}$ and

$$x_k \xrightarrow{w^*} x.$$

It follows that the sequence (x_k) is $\|.\|^*$-bounded.

Hence following Proposition A.3, we can suppose without any loss of generality that there exists a w^*-compact convex and metrisable subset K of E^*, such that

$$\forall a \in \Omega \quad \bigcup_n F_n(a) \subset K.$$

Hence

$$\mathrm{Ls}_n F_n(a) = \bigcap_n \mathrm{cl} \bigcup_{p\geqslant n} F_p(a).$$

Following Proposition A.3, the multifunction

$$a \mapsto \mathrm{Ls}_n F_n(a)$$

is graph measurable. This ends the proof of Claim A.4. □

A.4. **Komlós limits.** Let E be a separable Banach space and $(\Omega, \mathcal{A}, \mu)$ a finite measure space. A sequence (f_n) of mappings from Ω to E^* is said K-convergent to a mapping f, if for all subsequence (n_i) of (n)

$$\frac{1}{n}\sum_{i=1}^{n} f_{n_i}(a) \xrightarrow{\quad w^* \quad} f(a) \quad \text{a.e.}$$

Theorem A.2 (Komlós)**.** *Suppose that (φ_k) is a sequence of integrable real valued functions such that*

$$\sup_k \int_\Omega \mid \varphi_k \mid d\mu < +\infty.$$

Then there exists a subsequence (m) of (k) and an integrable real valued function φ such that (φ_m) is K*-convergent to φ.*

This theorem is due to Komlós [**?**].

A.5. **Gaposhkin.**

Lemma A.1 (Gaposhkin's lemma)**.** *Let E be a finite dimensional vector space and $(\Omega, \mathcal{A}, \mu)$ a finite measure space. If (f_n) is a mean norm bounded sequence of integrable mappings from Ω to E^*, then there exists a subsequence (n_k) of (n) such that for each $k \in \mathbb{N}$, $f_{n_k} = g_k + h_k$, where the sequence (g_k) is uniformly integrable and where the sequence (h_k) converges almost every where to 0.*

This Lemma is due to Gaposhkin, Lemma C.I in [**?**].

A.6. **Lyapunov.**

Theorem A.3 (Extended Lyapunov)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, let I be a finite set, let $\ell \in \mathbb{N}$, let $(f_i)_{i\in I}$ be measurable functions from $(\Omega, \mathcal{A}, \mu)$ to $\mathbb{R}^\ell$ and let $(\lambda_i)_{i\in I}$ measurable functions from Ω to $[0,1]$ with $\sum_{i\in I} \lambda_i(a) = 1$. Suppose that*

$$\int_\Omega \sum_{i\in I} \lambda_i(a)|f_i(a)| d\mu(a) < +\infty.$$

If $(\Omega, \mathcal{A}, \mu)$ *is non atomic then there exists a measurable partition* $(B_i)_{i \in I}$ *of* Ω *such that for each* $i \in I$*, the function* f_i *is integrable over* B_i *and*

$$\int_\Omega \sum_{i \in I} \lambda_i(a) f_i(a) d\mu(a) = \sum_{i \in I} \int_{B_i} f_i d\mu.$$

This theorem proved in Balder [**?**] is a corollary of the classical Lyapunov theorem.