Equilibrium Theory with a Measure Space of Possibly Satiated Consumers*

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Abstract

Walras equilibria may not exist when consumers' preferences are possibly satiated and to overcome this difficulty, several extended notions of equilibria have been proposed, which all reduce to Walras equilibria under nonsatiation and free disposal. This includes the notions of equilibria with slack (also called dividend equilibria) as in Drèze and Müller (1980), Makarov (1981), Aumann and Drèze (1986) and Mas-Colell (1992), monetary equilibria as in Kajii (1996), or weak equilibria as in Polemarchakis and Siconolfi (1993), which are all defined when there are finitely many consumers. This includes also the notion of free disposal equilibrium, when markets clear in a weak sense, allowing free disposal. Our paper considers an economy with a measure space of consumers and provides a general existence result of equilibria for the various existing notions. This result extends in particular the result by Hildenbrand (1970) on the existence of Walras equilibria.

Key words: equilibrium with slack, satiation, measure space of consumers, Walras equilibrium, excess demand, Fatou's Lemma.

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1 Introduction

Walras equilibria may not exist when consumers' preferences are possibly satiated. To overcome this difficulty, several extended notions of equilibria have been proposed, which all reduce to Walras equilibria under the nonsatiation and free disposal assumptions. These extended notions differ from Walras equilibrium, either by allowing some consumers to spend more than their standard Walrasian wealth, or by weakening the market clearing condition and allowing free disposal.

The first approach is related to the notions of equilibria with slack (also called dividend equilibria) by Drèze and Müller (1980), Makarov (1981), Aumann and Drèze (1986), and Mas-Colell (1992), and monetary equilibria as in Kajii (1996). Equilibria with slack formalize exactly the possibility of spending at equilibrium more than the Walrasian wealth (i.e., the value of the initial endowment plus the shares in the firms' profits), whereas Kajii (1996) interprets the slacks (or the dividends) as *paper money*, which is allocated to the consumers before the market takes place. We also mention the concept of weak equilibrium defined by Polemarchakis and Siconolfi (1993), where consumers are strived to maximize their preferences on budget sets that are defined by an equality constraint (instead of the standard inequality constraint). The second approach only differs from the Walrasian's one, by the fact that a weaker market clearing condition is allowed. More precisely, the total excess demand is required to be less than or equal to zero, for the order relation associated to a pointed, closed, convex cone C of $\mathbb{R}^H$. This takes into account the two important cases where $C = \{0\}$ and $C = \mathbb{R}^H_+$. This notion, which is often used as an intermediary concept to get the existence of equilibria under free disposal, will play here a more central role.

This paper considers an economy with a measure space of possibly satiated consumers. We extend to this framework the previous equilibrium notions, which were considered in economies with finitely many consumers and we focus on the existence problem. The justification of a measure space of consumers is discussed in Aumann (1965) and Hildenbrand (1974) and another motivation of the present paper is the validity of the core equivalence theorem without nonsatiation and free disposal. Before tackling this question, however, we need first to compare the equilibrium concepts, and clarify what are the "good" ones, taking for first and basic criterium of "goodness" that the existence problem can be solved. The situation with a measure space of consumers appears to be quite different from the case of finitely many consumers as it will be further discussed later.

Our main existence result [Theorem 3.1] will consider the model of a coalition production economy. We shall then consider a private ownership economy and provide in Theorem 3.2 conditions for the existence of quasi-equilibria with slack and free disposal quasi-equilibria. These two notions will be shown to play a complementary role in economies with a measure space of consumers . These two results generalize previous ones by Hildenbrand (1970), when we additionally assume local nonsatiation and free disposal. We also point out that the disposal cone C that we consider in the definition of free disposal equilibria (hence also in the free disposal assumption on the total production set, if any) may have an empty interior. The cone C is only assumed to be closed, convex and pointed, allowing us to provide a unified treatment of the two important cases $C = \{0\}$ and $C = \mathbb{R}^H_+$, and others. This leads us to provide a formulation of Fatou's lemma [Theorem B of the Appendix] to the case of a general disposal cone C.

The paper is organized as follows. In the following Section 2, we describe the models and present the various notions of equilibria. Section 2.1 will consider the model of the private ownership economy with a measure space of consumers and will define the notions of (quasi-)equilibria with slacks and free disposal (quasi-)equilibria. Section 2.2 will then consider the model of a coalition production economy with a measure space of consumers and will define the equilibrium notion in this framework. In Section 3, we present the main existence result [Theorem 3.1] in a coalition production economy and then deduce the existence result of quasi-equilibria [Theorem 3.2] in the context of a private ownership economy. Finally, in Section 4, we provide the proof of Theorem 3.1, and in the Appendix we state and prove the version of Fatou's lemma that is used in the existence proof.

2 The models and the equilibrium notions

2.1 Private ownership economies

We consider a private ownership economy $\mathcal{E}$ with finite sets H, J of commodities and producers, and a set A of consumers that will possibly be infinite. The commodity space[1] is $\mathbb{R}^H$ and an element x of $\mathbb{R}^H$ is called an action, or simply a consumption, or a production if the agent is, respectively, a consumer, a producer.

The set of consumers is assumed to be a positive finite complete measure space $(A, \mathcal{A}, \nu)$, where $\mathcal{A}$ is a σ-algebra of subsets of A, and ν is a σ-additive positive measure on $\mathcal{A}$ such that $\nu(A) = 1$. An element $E \in \mathcal{A}$ is a possible group of consumers, also called a coalition, and $\nu(E)$ represents the fraction of consumers which are in the coalition E. The first example of a measure space of consumers is the case originally considered by Aumann (1966), with $A = [0, 1]$, $\mathcal{A} = \mathcal{B}[0, 1]$ the σ-algebra of Borel subsets of $[0, 1]$ and ν the Lebesgue measure on $[0, 1]$. This framework encompasses also the case of a finite set A of consumers, by taking for $\mathcal{A}$ the set of all subsets of A and for ν the counting measure on A, defined by $\nu(E) = \sharp E/\sharp A$ for every $E \subset A$. The standard reference on economies with a measure space of consumers is Hildenbrand (1974).

Each consumer a is endowed with a consumption set $X(a) \subset \mathbb{R}^H$, a strict preference relation $\prec_a$ on $X(a)$ and an initial endowment of commodities $e(a) \in \mathbb{R}^H$. Given the price $p \in \mathbb{R}^H$, and the wealth $w \in \mathbb{R}$, the budget set of consumer a is defined by

$$B(a, p, w) = \{x \in X(a) \mid p \cdot x \leq w\}.$$

The preference relation $\prec_a$ will be assumed to be irreflexive and transitive[2], and the mapping $a \mapsto e(a)$, from A to $\mathbb{R}^H$, to be integrable. This allows us to define the total initial endowment of the economy to be $\int_A e(a) d\nu(a)$.

A consumption allocation specifies the possible consumptions of every consumer, hence is a mapping $x : A \to \mathbb{R}^H$ such that $x(a) \in X(a)$ for almost every (a.e.) $a \in A$ and it is further assumed to be integrable. We denote by $L^1(A, \mathbb{R}^H)$ the set of integrable mappings from A to $\mathbb{R}^H$ and by $\mathcal{X}$ the set of consumption allocations, which is thus

[1]We shall use the following notations. An element x of $\mathbf{R}^H$ is a mapping $x : H \to \mathbf{R}$ and it will also be denoted as $x = (x_h)_{h \in H}$, or simply $x = (x_h)$. This allows to have coordinate-free notations. Since the set H is finite, say of cardinal ℓ, identifying the finite dimensional space $\mathbf{R}^H$ with $\mathbf{R}^\ell$, we can also consider x as a vector $(x_1, \ldots, x_\ell)$ of $\mathbf{R}^\ell$. In $\mathbf{R}^H$ we denote by $x \cdot y := \sum_{h \in H} x_h y_h$, the scalar product, by $\|x\| := \sqrt{x \cdot x}$ the Euclidean norm, by $B(x_0, r) = \{x \in \mathbf{R}^H \mid \|x - x_0\| \leq r\}$, the closed ball, by $B = B(0, 1)$, the closed unit ball, and by $S = \{x \in \mathbf{R}^H \mid \|x\| = 1\}$, the unit sphere. The notation $x \geq y$ ($x >> y$) means that $x_h \geq y_h$ ($x_h > y_h$) for every $h \in H$ and we let $\mathbf{R}^H_+ = \{x \in \mathbf{R}^H \mid x \geq 0\}$ and $\mathbf{R}^H_{++} = \{x \in \mathbf{R}^H \mid x >> 0\}$. For $C \subset \mathbf{R}^H$ we denote by int C, cl C, co C, and C^+, respectively, the interior, the closure, the convex hull, and the positive polar cone of C, that is, $C^+ = \{x \in \mathbf{R}^H \mid x \cdot c \geq 0 \text{ for every } c \in C\}$. For $p \in \mathbf{R}^H$, we let $\sup p \cdot C = \sup\{p \cdot c \mid c \in C\}$.

[2]Following Schmeidler (1969), this allows us to consider preferences that may not be complete and encompasses the (classical) case where the tastes of each consumer are represented by a complete preordering $\preceq_a$ defined on $X(a)$ (hence also the case where the tastes are defined by a utility function), that is, $\preceq_a$ is a binary relation on $X(a)$, which is assumed to be reflexive, transitive and complete. In this case, the strict preference relation $\prec_a$ on $X(a)$ is defined, for x, x' in $X(a)$, by $x \prec_a x'$ if [$x \preceq_a x'$ and not $(x' \preceq_a x)$]. One easily sees that, if $\preceq_a$ is a complete preorder, then $\prec_a$ is irreflexive and transitive.

formally defined by :

$$\mathcal{X} = \left\{ x \in L^1(A, \mathbb{R}^H) \mid x(a) \in X(a) \text{ for a.e. } a \in A \right\}.$$

The production sector of the economy contains finitely many producers represented by their production sets $Y_j \subset \mathbb{R}^H$ ($j \in J$). The producers are owned by the consumers and the ownership shares of the consumers are given by integrable real-valued functions $\theta_j : A \to \mathbb{R}_+$ ($j \in J$) satisfying $\int_A \theta_j(a) d\nu(a) = 1$ for every $j \in J$.

The private ownership economy $\mathcal{E}$ is thus summarized by the list:

$$\mathcal{E} = \{\mathbb{R}^H, (A, \mathcal{A}, \nu), (X(a), \prec_a, e(a))_{a \in A}, (Y_j, \theta_j)_{j \in J}\}.$$

2.1.1 Equilibria with slack

This section presents the first notion, called equilibrium with slack (or with dividends), which was introduced with finitely many consumers by Drèze and Müller (1980), in a fixed-price setting, and then in a standard Arrow-Debreu model by Makarov (1981), Aumann and Drèze (1986) and Mas-Colell (1992).

Definition 2.1 *An element $(x^*, (y_j^*), p^*)$ in $\mathcal{X} \times (\mathbb{R}^H)^J \times \mathbb{R}^H$ is said to be an equilibrium (resp. quasi-equilibrium) with slack (or dividends) $d : A \to \mathbb{R}_+$ of the economy $\mathcal{E}$ if*

(a) [Preference Maximization] for a.e. $a \in A$, $x^*(a) \in B(a, p^*, w^*(a) + d(a))$, where

$$w^*(a) := p^* \cdot e(a) + \sum_{j \in J} \theta_j(a) \sup p^* \cdot Y_j \text{ denotes the Walrasian wealth}$$

and ($x \in X(a)$ and $x^*(a) \prec_a x$) imply $w^*(a) + d(a) < p^* \cdot x$;

[resp. ($x \in X(a)$ and $x^*(a) \prec_a x$) imply $w^*(a) + d(a) \leq p^* \cdot x$;]

(b) [Profit Maximization] for all $j \in J$, $y_j^* \in Y_j$ and $p^* \cdot y_j^* = \sup p^* \cdot Y_j$;

(c) [Market Clearing] $\int_A x^*(a) d\nu(a) = \int_A e(a) d\nu(a) + \sum_{j \in J} y_j^*$.

It is worth pointing out that the above definition does not require the (quasi-)equilibrium price to be nonzero as in the definition of a Walras (quasi-)equilibrium, which, with our notations, is exactly a (quasi-)equilibrium $(x^*, (y_j^*), p^*)$ whose slack d is null and whose equilibrium price p^* is non-zero.[3] In the following, we shall also say that $(x^*, (y_j^*), p^*)$ is an equilibrium with slack if we do not explicitly refer to the slack function $d : A \to \mathbb{R}_+$. One notices that every equilibrium with slack is clearly a quasi-equilibrium with slack.

Remark 2.1 Let us consider the important case, where almost every consumer a is locally nonsatiated, that is, for every $x \in X(a)$, and every neighborhood $\mathcal{N}$ of x, there exists $x' \in X(a) \cap \mathcal{N}$ such that $x \prec_a x'$. Let $(x^*, (y_j^*), p^*)$ be an arbitrary equilibrium with slack d. Then one easily shows that $d = 0$ and $p^* \neq 0$. In other words, the notions of equilibrium with slack and Walras equilibrium coincide when almost all the consumers are locally nonsatiated. We also point out that the same result does not hold in general under the weaker assumption of nonsatiation, even in the case of finitely many consumers.

[3]We can characterize (quasi-)equilibria with zero equilibrium prices and zero slacks as follows. The element $(x^*, (y_j^*), 0)$ is a quasi-equilibrium [resp. equilibrium] with zero slack if and only if $(x^*, (y_j^*))$ is an attainable allocation (i.e. $(x^*, (y_j^*)) \in \mathcal{X} \times \Pi_j Y_j$ satisfies the market clearing condition (iii)) [resp. $(x^*, (y_j^*))$ is an attainable allocation and for a.e. $a \in A$, $x^*(a)$ is a satiation point, i.e., there does not exist $x \in X(a)$ such that $x^*(a) \prec_a x$].

We end this section by noticing that the slack $d(a)$ in Definition 2.1 may be interpreted as the value of the initial endowment of an extra consumption commodity, called *paper money* in an extended economy. This economy can be defined as in Kajii (1996) in such a way that the preferences of the consumers and the production sets of the producers are not altered by *paper money*. Following the same arguments as in Kajii (1996), one can define a one-to-one correspondence between the equilibrium with slack of the initial economy and the Walras equilibrium of the extended economy.

2.1.2 Free disposal equilibria

We now present the second equilibrium notion considered in this article, which only differs from the Walrasian's one, by the weaker market clearing condition. More precisely, we impose that the total equilibrium excess demand be less than or equal to zero, for the order relation $\leq_C$ defined by a closed, convex cone C of $\mathbb{R}^H$, which is assumed to be pointed, that is, such that $C \cap (-C) = \{0\}$. We recall that the order relation $\leq_C$ is standardly defined by $x \leq_C y$ if and only if $y - x \in C$.

Definition 2.2 *A free disposal equilibrium (resp. quasi-equilibrium) of* $(\mathcal{E}, C)$ *is an element* $(x^*, (y_j^*), p^*)$ *in* $\mathcal{X} \times (\mathbb{R}^H)^J \times C^+$ *satisfying* $p^* \neq 0$*, together with*

(a) [Preference Maximization] for a.e. $a \in A$, $x^*(a) \in B(a, p^*, w^*(a))$*, where*

$$w^*(a) := p^* \cdot e(a) + \sum_{j \in J} \theta_j(a) \sup p^* \cdot Y_j \text{ denotes the Walrasian wealth}$$

and ($x \in X(a)$ *and* $x^*(a) \prec_a x$) *imply* $w^*(a) < p^* \cdot x$;

[resp. ($x \in X(a)$ *and* $x^*(a) \prec_a x$) *imply* $w^*(a) \leq p^* \cdot x$;]

(b) [Profit Maximization] for all $j \in J$, $y_j^* \in Y_j$ *and* $p^* \cdot y_j^* = \sup p^* \cdot Y_j$;

(c') *[C-Market Clearing]* $\int_A x^*(a) d\nu(a) - \int_A e(a) d\nu(a) - \sum_{j \in J} y_j^* \leq_C 0$.

When $C = \{0\}$, a free disposal (quasi-)equilibrium of $(\mathcal{E}, C)$ is exactly a Walras (quasi-)equilibrium of $\mathcal{E}$. We point out that, above, we did not assume the equilibrium value of the excess demand to be null, that is

$$p^* \cdot z^* = 0, \quad \text{where} \quad z^* = \int_A x^*(a) d\nu(a) - \int_A e(a) d\nu(a) - \sum_{j \in J} y_j^*.$$

One easily checks that this condition holds under the additional assumption that the consumers' preferences are locally nonsatiated.

2.2 Coalition production economies

Following Hildenbrand (1970), we consider the *coalition production economy*

$$\mathcal{E}_c = \{\mathbb{R}^H, (A, \mathcal{A}, \nu), (X(a), \prec_a, e(a), Y(a))_{a \in A}\},$$

where $Y(a) \subset \mathbb{R}^H$ denotes the production set available to consumer a and $Y_A := \int_A Y(a) d\nu(a)$[4] the total production set of the economy $\mathcal{E}_c$.

We now define the notion of a weak equilibrium and assume that C is a pointed, closed, convex cone of $\mathbb{R}^H$.

[4]That is, $\int_A Y(a) d\nu(a)$ is the integral of the correspondence $a \mapsto Y(a)$, which, we recall, is the set of all elements $\int_A f(a) d\nu(a)$, where $f : A \to \mathbb{R}^H$ is an integral mapping such that $f(a) \in Y(a)$ for a.e. $a \in A$.

Definition 2.3 *An element* (x^*, y^*, p^*) *in* $\mathcal{X} \times \mathbb{R}^H \times \mathbb{R}^H$ *is said to be a weak equilibrium of* $(\mathcal{E}_c, C)$ *if there exists a mapping* $d : A \to \mathbb{R}_+$ *such that:*
(a) [Preference Maximization] for a.e. $a \in A$, $x^*(a) \in B(a, p^*, w^*(a) + d(a))$, *where*

$$w^*(a) := p^* \cdot e(a) + \sup p^* \cdot Y(a) \text{ denotes the Walrasian wealth}$$

and ($x \in X(a)$ *and* $x^*(a) \prec_a x$) *imply* $w^*(a) + d(a) \leq p^* \cdot x$;

(b) [Profit Maximization] $y^* \in Y_A$ *and* $p^* \cdot y^* = \sup p^* \cdot Y_A$;

(c) [C-Market Clearing] $\int_A x^*(a) d\nu(a) - \int_A e(a) d\nu(a) - y^* \leq_C 0$.

We point out that we did not assume in the above definition that the weak equilibrium price p^* is nonzero. We also recall that the above notion coincides with Walras quasi-equilibrium (in the sense of Hildenbrand (1970)) when the slack d is zero, the market clearing condition is an equality, and the equilibrium price is nonzero.

Remark 2.2 Even if the above definition allows for slack and free disposal, the main aim of this article is to provide sufficient conditions which guarantee the existence of a weak equilibrium satisfying the additional condition

$$\text{if } C \neq \{0\} \text{ then } \|p^*\| = 1 \text{ and } d(a) = 0, \text{ for a.e. } a \in A. \qquad (\star)$$

In other words, for every weak equilibrium satisfying $(\star)$, the following alternative holds: either $C = \{0\}$ and the equilibrium market clearing condition is standard (i.e., it is an equality instead of an inequality), or $C \neq \{0\}$ and the budget constraint is standard (i.e., without slack).

2.3 An example

Consider an exchange economy $\mathcal{E}$ with two commodities, two consumers, having for consumption sets $X_1 = X_2 = \mathbb{R}^2_+$, for initial endowments $e_1 = (2, 2)$, $e_2 = (2, 2)$ and having the preferences represented by the following utility functions:

$$u_1(x_1, x_2) = -\|(x_1, x_2) - (1, 1)\|^2 \quad \text{and} \quad u_2(x_1, x_2) = x_1 + x_2.$$

This economy does not admit any Walras equilibrium. Indeed, let (x_1^*, x_2^*, p^*) be a Walrasian equilibrium, then it is easy to see that the first consumer will choose $x_1^* = (1, 1)$, hence $x_2^* = (3, 3)$, but $(3, 3)$ does not maximize the utility of the second consumer under her budget constraint.

However, the economy $\mathcal{E}$ has an equilibrium with slack (x_1^*, x_2^*, p^*) defined by $x_1^* = (1, 1)$, $x_2^* = (3, 3)$, $p^* = (1, 1)$ for the slacks $d_1 = 0, d_2 = 2$. We can interpret this equilibrium with slack as a Walras equilibrium in the extended economy as in Kajii(1996), where the initial endowment of *paper money* is 0 unit for the first consumer and 2 units for the second one. Then, the extended economy admits a Walras equilibrium $(\hat{x}_1^*, \hat{x}_2^*, \hat{p}^*)$ defined by $\hat{x}_1^* = (2, 1, 1)$, $\hat{x}_2^* = (0, 3, 3)$, $\hat{p}^* = (1, 1, 1)$. Thus, the first consumer collects all the *paper money* poured in the economy, and (initially) allocated to the second consumer.

For the cone $C = \mathbb{R}^2_+$, the exchange economy $\mathcal{E}$ admits also a free disposal equilibrium $(x_1^{**}, x_2^{**}, p^{**})$, defined by $x_1^{**} = (1, 1)$, $x_2^{**} = (2, 2)$, $p^{**} = (1, 1)$. It is worth pointing out that the allocation (x_1^{**}, x_2^{**}) is Pareto dominated by the allocation (x_1^*, x_2^*). Finally, we mention the equilibrium notion introduced by Polemarchakis and Siconolfi (1993) in an exchange economy without free disposal and nonsatiation (and also called "weak equilibrium"). They weaken the Walrasian equilibrium notion by imposing that each consumer is maximizing her preferences on the budget set defined with the equality constraint (instead of the inequality constraint). One easily sees that the exchange economy $\mathcal{E}$ admits an equilibrium $(\bar{x}_1, \bar{x}_2, \bar{p})$, in the sense of Polemarchakis-Siconolfi, defined by $\bar{x}_1 = (2,2)$, $\bar{x}_2 = (2,2)$, $\bar{p} = (1,1)$. Again, it is worth pointing out that the weak equilibrium allocation $(\bar{x}_1, \bar{x}_2)$ is Pareto dominated by the allocation (x_1^*, x_2^*), and we refer to Kajii (1996) for further discussions on the welfare properties of equilibria.

3 Existence results

3.1 Coalition production economies

3.1.1 The main existence result

We posit the following assumptions on the coalition production economy $\mathcal{E}_c$.

Assumption $\mathbf{M}_c$ [Measurability] (i) The consumption set correspondence $a \mapsto X(a)$, the preference correspondence $a \mapsto \prec_a$, and the production set correspondence $a \mapsto Y(a)$ are measurable[5]; (ii) the mapping $e(.) : A \to \mathbb{R}^H$ is integrable.

Assumption C [Consumption side] For a.e. $a \in A$, $X(a)$ is closed, and the strict preference relation $\prec_a$ is irreflexive, transitive, continuous and convex on every atom of A.[6]

Assumption $\mathbf{P}_c$ [Production Side] Y_A is nonempty, closed and convex.

Assumption $\mathbf{B}_c$ [Boundedness] There exists a pointed closed convex cone C in $\mathbb{R}^H$ such that (i) the correspondence $a \mapsto X(a)$ is C-integrably bounded, in the sense that, for a.e. $a \in A$, $X(a) \subset B(0, \rho(a)) + C$, for some integrable function $\rho : A \to \mathbb{R}_+$, and (ii) $\mathbf{A}(Y_A) \cap C = \{0\}$, where $\mathbf{A}(Y_A)$ denotes the asymptotic cone[7] of Y_A.

Assumption $\mathbf{S}_c$ [Survival] $e(a) \in X(a) - Y(a)$ for a.e. $a \in A$.

We now state the main result of the paper.

Theorem 3.1 *Let $\mathcal{E}_c$ be a coalition production economy satisfying Assumptions $\mathbf{M}_c$, $\mathbf{C}$, $\mathbf{P}_c$, $\mathbf{S}_c$, and $\mathbf{B}_c$. Then, for every nonnegative integrable function $\delta : A \to \mathbb{R}_+$, the economy $(\mathcal{E}_c, C)$ admits a weak equilibrium (x^*, y^*, p^*) whose slack d satisfies:*

$$d(a) = \delta(a)(1 - \|p^*\|) \text{ for a.e. } a \in A,$$

and such that, if $C \neq \{0\}$, then $\|p^*\| = 1$ (hence $d(a) = 0$ for a.e. $a \in A$).

The proof of Theorem 3.1 is given in Section 4.

[5] A correspondence F from A to $\mathbf{R}^n$ is said to be measurable if its graph $G(F) := \{(a, x) \in A \times \mathbf{R}^n \mid x \in F(a)\}$ belongs to $\mathcal{A} \otimes \mathcal{B}(\mathbf{R}^n)$, where $\mathcal{B}(\mathbf{R}^n)$ denotes the σ-algebra of Borel subsets of $\mathbf{R}^n$. The preference correspondence $a \mapsto \prec_a$ is measurable in the sense that the correspondence $a \mapsto \{(x, x') \in X(a) \times X(a) \mid x \prec_a x'\}$ is measurable.

[6] That is, for every x_1, x_2, x_3 in $X(a)$, [irreflexive] not $[x_1 \prec_a x_1]$, [transitive] $[x_1 \prec_a x_2$ and $x_2 \prec_a x_3]$ implies $x_1 \prec_a x_3$, [continuous] for every $x \in X(a)$, the sets $\{z \in X(a) \mid z \prec_a x\}$ and $\{z \in X(a) \mid x \prec_a z\}$ are open in $X(a)$ (for its relative topology), and [convex on every atom] for every atom E of A and a.e. $a \in E$, $X(a)$ is convex and $\prec_a$ is convex, that is, for every $x \in X(a)$, the set $\{z \in X(a) \mid \text{not}[z \prec_a x]\}$ is convex. We recall that an element $E \in \mathcal{A}$ is said to be an atom of the measure space $(A, \mathcal{A}, \nu)$ if $\nu(E) \neq 0$ and $[B \in \mathcal{A}$ and $B \subset E]$ implies $\nu(B) = 0$ or $\nu(E \setminus B) = 0$.

[7] We recall that for a convex set $Y \subset \mathbf{R}^H$, one can define the asymptotic cone of Y as $\mathbf{A}Y := \{v \in \mathbf{R}^H \mid Y + v \subset Y\}$ [Theorem 8.1 of Rockafellar (1970)] and we refer to Debreu (1959) for an alternative definition in the general (non-convex) case.

3.1.2 Hildenbrand's result on the existence of quasi-equilibria

Assuming in addition local nonsatiation and free disposal, we now give a direct consequence of Theorem 3.1. In the following, we assume that $C \neq \{0\}$, since the local nonsatiation Assumption is not compatible with integrably bounded consumption sets (the case $C = \{0\}$) [8]

Corollary 3.1 *The economy $\mathcal{E}_c$ admits a Walras quasi-equilibrium if it satisfies Assumptions $\mathbf{M}_c$, $\mathbf{C}$, $\mathbf{P}_c$, $\mathbf{S}_c$, $\mathbf{B}_c$, $C \neq \{0\}$ and the following hold*

Assumption FD [Free Disposal] $Y_A - C \subset Y_A$;

Assumption LN [Local Nonsatiation] *for a.e. $a \in A$, for every consumption $x \in X(a)$ and every neighborhood $\mathcal{N}$ of x, there is $x' \in X(a) \cap \mathcal{N}$ such that $x \prec_a x'$.*

The above corollary is essentially the existence result in Hildenbrand (1970), with some slight generalizations that we now comment. First, we only assume the consumption set $X(a)$ to be closed for a.e. consumer a, without any convexity assumption as in Hildenbrand (1970). Allowing nonconvex consumption sets is of fundamental importance to consider models of spatial economies as in Cornet and Médecin (2000), in which nonconvexities arise structurally. Note that this is only a byproduct of our study and that this is only true for quasi-equilibria and not for equilibria (see the work of Yamazaki (1978) in this case). Second, Hildenbrand (1970) is considering the disposal cone $C = \mathbb{R}^H_+$ whereas, in our Assumption $\mathbf{B}_c$ the disposal cone C is only assumed to be closed, convex and pointed, hence may have an empty interior.

Proof. From Theorem 3.1, the economy $\mathcal{E}_c$ admits a weak equilibrium (x^*, y^*, p^*) with $\|p^*\| = 1$ (since $C \neq \{0\}$). From the Local Nonsatiation Assumption, each consumer a is binding her budget constraint, hence

$$p^* \cdot x^*(a) = p^* \cdot e(a) + \sup p^* \cdot Y(a) \quad \text{for a.e.} \quad a \in A.$$

Integrating over A one deduces that

$$p^* \cdot z^* = 0, \quad \text{where} \quad z^* = \int_A x^*(a) d\nu(a) - \int_A e(a) d\nu(a) - y^* \in C.$$

But $y^* - z^* \in Y_A - C \subset Y_A$ from the Free Disposal Assumption, hence, there exists $y^{**} \in Y_A$ such that $y^* - z^* = y^{**}$ and one has $\sup p^* \cdot Y_A = p^* \cdot y^* = p^* \cdot y^{**}$. One can easily check that (x^*, y^{**}, p^*) is a Walras quasi-equilibrium of $\mathcal{E}_c$. ∎

We end this section with a discussion of the Boundedness Assumption $\mathbf{B}_c$, by giving several possible choices of the cone C. The first case $C = \{0\}$ is excluded in this section since $\mathbf{B}_c$ reduces to the fact that the correspondence $a \mapsto X(a)$ is integrably bounded. By taking $C = \mathbb{R}^H_+$, we see that Assumption $\mathbf{B}_c$ is slightly weaker than the following assumption made by Hildenbrand (1970):

Assumption B$_1$ (i) The correspondence $a \mapsto X(a)$ is integrably bounded below, in the sense that, for a.e. $a \in A$, $X(a) \subset \{\underline{x}(a)\} + \mathbb{R}^H_+$, where $\underline{x} : A \to \mathbb{R}^H$ is integrable; (ii) $Y_A \cap \mathbb{R}^H_+ = \{0\}$; (iii) $Y_A \cap -Y_A = \{0\}$.

We also point out Assumption $\mathbf{B}_c$ is a consequence of the following synthetic and self-explanatory assumption (take $C = -\mathbf{A}(Y_A)$):

Assumption B$_2$ (i) for a.e. $a \in A$, $X(a) \subset B(0, \rho(a)) - \mathbf{A}(Y_A)$, where $\rho : A \to \mathbb{R}_+$ is integrable; (ii) $\mathbf{A}(Y_A) \cap -\mathbf{A}(Y_A) = \{0\}$,

which is weaker that the above assumption $\mathbf{B}_1$ under the free disposal assumption.

[8] Indeed, the classical argument follows if there exists a continuous utility function representing the strict preorder $\prec_a$. A direct argument can also be given as in Cornet and als (2002).

3.2 Private ownership economies

Let $\mathcal{E}$ be a private ownership economy. We denote by $Y := \sum_j Y_j$ the total production set of $\mathcal{E}$ and we posit the following assumptions:

Assumption M (i) The consumption set correspondence $a \mapsto X(a)$ and the preference correspondence $a \mapsto \prec_a$ are measurable; (ii) for every $j \in J$, the function $\theta_j : A \to \mathbb{R}_+$ is integrable and $\int_A \theta_j(a) d\mu(a) = 1$; (iii) the mapping $e(.) : A \to \mathbb{R}^H$ is integrable.

Assumption S $e(a) \in X(a) - \sum_j \theta_j(a) \overline{co} Y_j - \mathbf{A}Y$ for a.e. $a \in A$.[9]

We denote by **P** and **B** the assumptions obtained from $\mathbf{P}_c$ and $\mathbf{B}_c$, replacing the total production set Y_A of $\mathcal{E}_c$ by the total production Y of $\mathcal{E}$. We now state the following existence result.

Theorem 3.2 *Let $\mathcal{E}$ be a private ownership economy satisfying Assumptions* **M**, **C**, **P**, **S**, *and* **B**.
(i) If $C = \{0\}$, for every nonnegative integrable function $\delta : A \to \mathbb{R}_+$, the economy $\mathcal{E}$ admits a quasi-equilibrium with slack $(x^*, (y_j^*), p^*)$, such that $\|p^*\| \leq 1$ whose slack d satisfies:

$$d(a) = \delta(a)(1 - \|p^*\|) \quad \textit{for a.e.} \quad a \in A.$$

(ii) If $C \neq \{0\}$, the economy $(\mathcal{E}, C)$ admits a free disposal quasi-equilibrium.

Proof. It is a direct consequence of Theorem 3.1. We associate to the private ownership economy $\mathcal{E}$ the coalition production economy $\mathcal{E}_c$, whose characteristics are the same as for $\mathcal{E}$, but the production sets $Y(a)$, which are defined by

$$Y(a) := \sum_{j \in J} \theta_j(a) \overline{co} Y_j + \mathbf{A}Y.$$

From Hildenbrand (1970), the two economies $\mathcal{E}$ and $\mathcal{E}_c$ have the same total production set, i.e., $Y_A = \sum_{j \in J} Y_j$. Consequently, if $\mathcal{E}$ satisfies Assumptions **M**, **C**, **P**, **B** and **S**, then $\mathcal{E}_c$ satisfies Assumptions $\mathbf{M}_c$, **C**, $\mathbf{P}_c$, $\mathbf{B}_c$ and $\mathbf{S}_c$. From Theorem 3.1 there exists a weak equilibrium (x^*, y^*, p^*) of $\mathcal{E}_c$. But $y^* \in Y_A = \sum_{j \in J} Y_j$, hence $y^* = \sum_{j \in J} y_j^*$ for some $y_j^* \in Y_j$ $(j \in J)$. Recalling that $\sup p^* \cdot Y(a) = \sum_{j \in J} \theta_j(a) \sup p^* \cdot Y_j$ (see Hildenbrand (1970) for the proof), one easily checks that $(x^*, (y_j^*), p^*)$ is a quasi-equilibrium with slack of $\mathcal{E}$, if $C = \{0\}$, and a free disposal quasi-equilibrium of $(\mathcal{E}, C)$, if $C \neq \{0\}$. ∎

In the case of finitely many consumers, the existence of a quasi-equilibrium with slack is also proved when $C \neq \{0\}$ (see Mas-Colell (1992), Kajii (1996)). We refer to Remark 4.1 below, which explains why the proof given in this paper carries to the case $C \neq \{0\}$ only when A is finite.

We now give two consequences of Theorem 3.2. The first one provides an existence result of equilibria with slack and free disposal equilibria. The proof is standard and we refer to Cornet and als (2002) for details.

Corollary 3.2 *Let $\mathcal{E}$ be a private ownership economy such that* **M**, **C**, **P**, **B** *hold, for a.e. $a \in A$, $X(a)$ is convex, and assume that:*

Assumption SS [Strong Survival][10]

[9] This assumption is implied in particular by the following (more standard) one:(i) for a.e. $a \in A$, $e(a) \in X(a) - \mathbf{A}Y$, and (ii) for every $j \in J$, $0 \in Y_j$.

[10] This assumption is implied in particular by the following (more standard) one:(i) for a.e. $a \in A$, $e(a) \in X(a) -$ int $[\mathbf{A}Y]$, and (ii) for every $j \in J$, $0 \in Y_j$.

$e(a) \in \text{ int } [X(a) - \sum_{j \in J} \theta_j(a) \overline{\text{co}}\, Y_j - \mathbf{A}Y]$ *for a.e.* $a \in A$.

(i) If $C = \{0\}$*, for every positive integrable function* $\delta : A \to \mathbb{R}_{++}$*, the economy* $\mathcal{E}$ *admits an equilibrium with slack* $(x^*, (y_j^*), p^*)$*, such that* $\|p^*\| \leq 1$ *whose slack* d *satisfies:* $d(a) = \delta(a)(1 - \|p^*\|)$ *for a.e.* $a \in A$.

(ii) If $C \neq \{0\}$*, the economy* $(\mathcal{E}, C)$ *admits a free disposal equilibrium.*

The next one provides an existence result of Walras quasi-equilibria, which is mainly a result by Hildenbrand (1970), as discussed previously. The proof of this last result is a simple adaptation of the proof of Corollary 3.1.

Corollary 3.3 *The economy* $\mathcal{E}$ *admits a Walras quasi-equilibrium if it satisfies Assumptions* **M**, **C**, **P**, **S**, **B**, $C \neq \{0\}$ *and the following hold*

Assumption FD [Free Disposal] $Y - C \subset Y$;

Assumption LN [Local Nonsatiation] *for a.e.* $a \in A$*, for every consumption* $x \in X(a)$ *and every neighborhood* $\mathcal{N}$ *of* x*, there is* $x' \in X(a) \cap \mathcal{N}$ *such that* $x \prec_a x'$.

4 Proof of Theorem 3.1

The proof of Theorem 3.1 proceeds in several steps. The first step [Section 4.1] associates to the coalition production economy $\mathcal{E}_c$, and to every integer k, a "truncated" economy $\mathcal{E}_c^k$, whose consumption sets and production sets are bounded. The second step [Section 4.2] shows that the excess demand correspondence Z^k of the "truncated" economy $\mathcal{E}_c^k$ is upper semi-continuous (u.s.c.) with nonempty, convex, compact values and satisfies Walras' law. The third step [Section 4.3] associates to the economy $\mathcal{E}_c^k$, via a fixed-point argument, a list $(x^k, y^k, p^k) \subset \mathcal{X} \times Y_A \times B$. The fourth step [Section 4.4] shows that the sequence $\{(\int_A x^k(a) d\nu(a), y^k, p^k)\} \subset \mathbb{R}^H \times Y_A \times B$ is bounded. Hence, by Fatou's Lemma [Theorem B of the Appendix] it admits a subsequence converging to some element $(\xi^*, y^*, p^*) \in \mathbb{R}^H \times Y_A \times B$ and $\int_A x^*(a) d\nu(a) \leq_C \xi^*$ for some $x^* \in \mathcal{X}$. Finally, it will be shown that (x^*, y^*, p^*) is a weak equilibrium of the coalition production economy $\mathcal{E}_c$.

4.1 Truncation of the economy

From the Survival Assumption $\mathbf{S}_c$, there exist two mappings $\hat{x}$ and $\hat{y}$, from A to $\mathbb{R}^H$, such that for a.e. $a \in A$, $\hat{x}(a) \in X(a)$, $\hat{y}(a) \in Y(a)$, and $\hat{x}(a) = e(a) + \hat{y}(a)$. The following lemma guarantees that the mappings $\hat{x}$ and $\hat{y}$ can be chosen to be integrable.

Lemma 4.1 *There exist two integrable mappings* $\hat{x}$ *and* $\hat{y}$*, from* A *to* $\mathbb{R}^H$*, such that*

$$\textit{for a.e. } a \in A,\ \hat{x}(a) \in X(a),\ \hat{y}(a) \in Y(a),\ \textit{and}\ \hat{x}(a) = e(a) + \hat{y}(a).$$

Proof. From the Survival Assumption $\mathbf{S}_c$, one deduces that

$$\text{for a.e. } \ a \in A, \quad M(a) := \{(x, y) \in X(a) \times Y(a) \mid x = e(a) + y\} \neq \emptyset.$$

Furthermore, from the Measurability Assumption $\mathbf{M}_c$, the correspondence $a \mapsto M(a)$ is easily shown to be measurable. Hence, by Aumann's measurable selection theorem (see Aumann (1965)), there exist two measurable mappings $\hat{x}, \hat{y}$, from A to $\mathbb{R}^H$, such that, for a.e. $a \in A$, $(\hat{x}(a), \hat{y}(a)) \in M(a)$.

We end the proof by showing that the mapping $\hat{y}$ is integrable, which clearly implies that $\hat{x}$ is also integrable, since $e(\cdot)$ is integrable by Assumption $\mathbf{M}_c$. From Assumption

$\mathbf{P}_c$, the set $Y_A := \int_A Y(a)d\nu(a)$ is nonempty, hence there exists an integrable mapping $y: A \to \mathbb{R}^H$ such that $y(a) \in Y(a)$ for a.e. $a \in A$. For all integer k, we define the mapping $f^k: A \to \mathbb{R}^H$ by

$$f^k := \chi_{A^k}\hat{y} + \chi_{A\setminus A^k}y, \quad \text{where } A^k := \{a \in A \mid \|\hat{y}(a)\| \leq k\},$$

and χ_S is standardly defined by $\chi_S(a) = 1$ if $a \in S$ and $\chi_S(a) = 0$ if $a \notin S$. Clearly, the mappings f^k are integrable and satisfy the following three properties: (i) for a.e. $a \in A$, $\lim_{k\to\infty} f^k(a) = \hat{y}(a)$; (ii) for every k and for a.e. $a \in A$, $f^k(a) \in B(0, g(a)) + C$, where $g(a) = \max\{\rho(a) + \|e(a)\|, \|y(a)\|\}$, and ρ is defined by Assumption $\mathbf{B}_c$, hence the function $g : A \to \mathbb{R}_+$ is integrable; (iii) for every k, $\int_A f^k(a)d\nu(a)$ belongs to $Y_A \cap [B(0, \int_A g(a)d\nu) + C]$, which is bounded, since its asymptotic cone $\mathbf{A}(Y_A) \cap C$ is reduced to $\{0\}$, by Assumption $\mathbf{B}_c$. From Assertion (iii), without any loss of generality, we can assume that the sequence $\{\int_A f^k(a)d\nu(a)\}$ is convergent. Consequently, from Fatou's lemma [Theorem **B** in Appendix], there exists an integrable mapping $f : A \to \mathbb{R}^H$, such that, for a.e. $a \in A$, $f(a)$ is adherent to the sequence $\{f^k(a)\}$. But, from the above Assertion (i), for a.e. $a \in A$, the sequence $\{f^k(a)\}$ converges to $\hat{y}(a)$. Hence $\hat{y}$ coincides with the integrable mapping f almost everywhere and is also integrable. ■

For every integer k, we define the truncated economy

$$\mathcal{E}^k = \{\mathbb{R}^H, (A, \mathcal{A}, \nu), (X^k(a), \prec_a, e(a), Y^k(a))_{a\in A}\},$$

which differs only from $\mathcal{E}_c$, by the choice of the consumption sets $X^k(a)$ and the production sets $Y^k(a)$, which are defined as follows:

$$X^k(a) := X(a) \cap B(0, \max\{k, \|\hat{x}(a)\|\}),$$

$$Y^k(a) := Y(a) \cap B(0, \max\{k, \|\hat{y}(a)\|\}),$$

where $\hat{x}$ and $\hat{y}$ are two integrable mappings, chosen as in Lemma 4.1. For all $(a, p, w) \in A \times \mathbb{R}^H \times \mathbb{R}$, we let the budget set and the "quasi-demand" of consumer a be defined as follows:

$$B^k(a, p, w) := \{x \in X^k(a) \mid p \cdot x \leq w\}$$

$$D^k(a, p, w) := \{x \in B^k(a, p, w) \mid \text{ there is no } x' \in X^k(a),\ p \cdot x' < w \text{ and } x \prec_a x'\}.$$

In the economy $\mathcal{E}_c^k$, adapting Bergstrom (1976), the wealth $r^k(a, p)$ of consumer a, the total quasi-demand D^k, the total production set Y_A^k, the total supply S^k, and the excess demand Z^k are defined respectively as follows:

$$\begin{aligned}
r^k(a,p) &:= p \cdot e(a) + \sup p \cdot Y^k(a) + \delta(a)(1 - \|p\|), \\
D^k(p) &:= \int_A D^k(a, p, r^k(a,p))d\nu(a), \\
Y_A^k &:= \int_A Y^k(a)d\nu(a), \\
S^k(p) &:= \{y \in \mathrm{cl}(\mathrm{co}Y_A^k) \mid p \cdot y = \sup p \cdot \mathrm{cl}(\mathrm{co}Y_A^k)\}, \\
Z^k(p) &:= D^k(p) - \{\int_A e(a)d\nu(a)\} - S^k(p).
\end{aligned}$$

4.2 The excess demand of the truncated economy

The main property of the excess demand correspondence Z^k of $\mathcal{E}_c^k$ is stated below.

Proposition 4.1 *Under Assumptions* $\mathbf{M}_c$*,* $\mathbf{C}$*,* $\mathbf{P}_c$*,* $\mathbf{B}_c$ *and* $\mathbf{S}_c$*, the correspondence* Z^k *from* $C^+ \cap B$ *to* $\mathbb{R}^H$ *is upper semi-continuous (u.s.c.) with nonempty, convex, compact values and satisfies Walras' law, i.e., for every* $p \in C^+ \cap S$, $\sup p \cdot Z^k(p) \leq 0$.

Proof. *Step 1:* Z^k *is u.s.c. with nonempty, convex, compact values.* It is sufficient to show that both correspondences S^k and D^k are u.s.c. with nonempty, convex, compact values. As for S^k, the proof is a standard consequence of the maximum theorem (see Berge (1959)) and the fact that the set $\mathrm{cl}(\mathrm{co}Y_A^k)$ is nonempty, convex and compact. We shall deduce the same property for D^k from the following result.

Theorem A *Let* $[\mathbb{R}^H, (A, \mathcal{A}, \nu), (\bar{X}(a), \prec_a, e(a))_{a\in A}]$ *satisfy assumptions* $\mathbf{M}_c$ *and* $\mathbf{C}$*, assume in addition that the correspondence* $a \mapsto \bar{X}(a)$ *is integrably bounded, that is,* $\bar{X}(a) \subset B(0, \rho(a))$ *for some integrable function* $\rho : A \to \mathbb{R}_+$*, and let* $\bar{r} : A \times \mathbb{R}^H \to \mathbb{R}$ *be a Caratheodory function, that is, for every* $p \in \mathbb{R}^H$*, the function* $a \mapsto \bar{r}(a, p)$ *is measurable and, for every* $a \in A$*, the function* $p \mapsto \bar{r}(a, p)$ *is continuous. We let*

$$P := \{p \in \mathbb{R}^H \mid \inf p \cdot \bar{X}(a) \leq \bar{r}(a, p) \text{ for a.e. } a \in A\}.$$

Then the quasi-demand correspondence $\bar{D}$*, from* P *to* $\mathbb{R}^H$*, is u.s.c. with nonempty, compact, convex values and takes its values in a fixed compact set* $K \subset \mathbb{R}^H$*, i.e., for every* $p \in P$*,* $\bar{D}(p) \subset K$*.*

The above result is essentially due to Hildenbrand (1970) with two slight differences. In Theorem A, we do not assume that the preferences are complete and that the consumption sets $X(a)$ are convex. Nevertheless, the proof of Theorem A is a simple adaptation of the one in Hildenbrand (1970).

We now come back to the proof of Proposition 4.1. Applying Theorem A to $\bar{X}(a) = X^k(a)$ and $\bar{r} = r^k$, we immediately check that Assumptions $\mathbf{M}_c$ and $\mathbf{C}$ are satisfied, that the correspondence $a \mapsto X^k(a)$ is integrably bounded (from the definition of $X^k(a)$ and Lemma 4.1), and that r^k is a Caratheodory function (using the compactness of $Y^k(a)$). The proof will thus be complete if we show that

$$C^+ \cap B \subset \{p \in \mathbb{R}^H \mid \inf p \cdot X^k(a) \leq r^k(a, p) \text{ for a.e. } a \in A\}.$$

Indeed, recalling that, for a.e. $a \in A$, $\hat{x}(a) \in X^k(a)$, $\hat{y}(a) \in Y^k(a)$, and $\hat{x}(a) = \hat{y}(a) + e(a)$ (from the definitions of the sets $X^k(a)$ and $Y^k(a)$ and Lemma 4.1), for every $p \in C^+ \cap B$, one gets:

$$\begin{aligned} r^k(a,p) &:= p \cdot e(a) + \sup p \cdot Y^k(a) + \delta(a)(1 - \|p\|) \\ &\geq p \cdot e(a) + p \cdot \hat{y}(a) + 0 = p \cdot \hat{x}(a) \geq \inf p \cdot X^k(a). \end{aligned}$$

Step 2: Z^k *satisfies Walras' law.* Indeed, for all $p \in C^+ \cap S$, let $z \in Z^k(p)$, i.e., $z = \int_A x(a) d\nu(a) - \int_A e(a) d\nu(a) - y$, for some integrable mapping $x : A \to \mathbb{R}^H$ such that for a.e. $a \in A$, $x(a) \in D^k(a, p, r^k(a, p))$, and for some $y \in S^k(p)$. From the definitions of the correspondence D^k and S^k, for a.e. $a \in A$ and every $p \in C^+ \cap S$ (hence $\|p\| = 1$), one has

$$p \cdot x(a) \leq r^k(a, p) := p \cdot e(a) + \sup p \cdot Y^k(a) + 0.$$

Recalling the following equality (see Hildenbrand (1974))

$$\int_A \sup p \cdot Y^k(a) d\nu(a) = \sup p \cdot \int_A Y^k(a) d\nu(a) = \sup p \cdot Y^k_A$$

and the fact that $\sup p \cdot Y^k_A = \sup p \cdot \mathrm{cl}(\mathrm{co} Y^k_A)$, integrating on A the above inequality, we deduce that

$$p \cdot \int_A x(a) d\nu(a) \leq p \cdot \int_A e(a) d\nu(a) + \sup p \cdot \mathrm{cl}(\mathrm{co} Y^k_A) = p \cdot \int_A e(a) d\nu(a) + p \cdot y,$$

which implies that $p \cdot z \leq 0$. Hence, Z^k satisfies Walras' law. ∎

4.3 The fixed point argument

The key tool to prove the existence of equilibria is the following version of Debreu-Gale-Nikaido's lemma (see Gale (1955), Nikaido (1956), Debreu (1956)).

Theorem 4.1 *Let Q be a closed convex cone of $\mathbb{R}^H$, $Q \neq \{0\}$, and let Z be an u.s.c. correspondence from $Q \cap B$ to $\mathbb{R}^H$ with nonempty, convex, compact values, satisfying Walras' law, i.e., for every $p \in Q \cap S$, $\sup p \cdot Z(p) \leq 0$. Then there exists $p^* \in Q \cap B$ such that $Z(p^*) \cap -Q^+ \neq \emptyset$ and if Q is not a subspace we can further choose p^* to be in S.*

Proof. We let $\Delta := Q \cap B$. Since Z is u.s.c. with compact values and Δ is compact, the set $\cup_{p \in \Delta} Z(p)$ is compact (see Berge (1959)), hence is contained in some nonempty, convex, compact subset K of $\mathbb{R}^H$. We now consider the correspondence F from $\Delta \times K$ to itself defined by :

$$F(p, z) = \{\overline{q} \in \Delta \mid \overline{q} \cdot z \geq q \cdot z, \ \forall q \in \Delta\} \times Z(p).$$

One easily checks that F is an u.s.c. correspondence with nonempty, convex, compact values. Consequently, from Kakutani's theorem, the correspondence F has a fixed point (p^*, z^*) in $\Delta \times K$, that is,

$$z^* \in Z(p^*), \text{ and } \ p^* \cdot z^* \geq q \cdot z^* \text{ for all } \ q \in \Delta. \tag{1}$$

We now show that $p^* \cdot z^* \leq 0$. Indeed, if $p^* = 0$, this is clearly true, and if $\|p^*\| = 1$, from (1) and Walras' law, we get $p^* \cdot z^* \leq \sup p^* \cdot Z(p^*) \leq 0$. Assume now that $p^* \neq 0$ and $\|p^*\| < 1$, then $p^*/\|p^*\| \in \Delta$. From the above condition (1), $p^* \cdot z^* \geq (p^*/\|p^*\|) \cdot z^*$, hence $(1 - \|p^*\|) p^* \cdot z^* \leq 0$. This implies that $p^* \cdot z^* \leq 0$.

From the above condition (1), we deduce that $q \cdot z^* \leq p^* \cdot z^* \leq 0$ for every $q \in \Delta$, and clearly for every $q \in Q$ we still have $q \cdot z^* \leq 0$. Consequently, $-z^* \in Q^+$, which ends the proof of the first part.

We now show that we can additionally choose $p^* \in S$ when Q is not a subspace. This is exactly the statement of Debreu (1956), applied to the restriction of the correspondence Z to $Q \cap S$, which is clearly u.s.c. with nonempty, convex, compact values and satisfies Walras' law on $Q \cap S$. ∎

In view of Proposition 4.1, the assumptions of the above fixed-point theorem are satisfied by the excess demand correspondence of the economy $\mathcal{E}^k_c$, i.e., by $Z := Z^k$ and by the closed, convex cone $Q = C^+$, where C is given by Assumption $\mathbf{B}_c$. Indeed, the cone Q is different from $\{0\}$ since C is pointed. Consequently, from Theorem 4.1, for every k, there exists $p^k \in C^+ \cap B$ such that $Z^k(p^k) \cap -C^{++} \neq \emptyset$. Furthermore, $C^{++} = C$ from the bipolar theorem (see, for example, Rockafellar (1970)) since C is a closed, convex cone. Thus there exists a sequence $\{(x^k, y^k, z^k, p^k)\} \subset \mathcal{X} \times Y_A^k \times (-C) \times [C^+ \cap B]$ such that:

$$\text{for a.e. } a \in A, \; x^k(a) \in D^k(a, p^k, r^k(a, p^k)); \tag{2}$$

$$y^k \in S^k(p^k), \;\; \text{i.e.,} \;\; y^k \in \text{cl}(\text{co}Y_A^k) \;\; \text{and} \;\; p^k \cdot y^k = \sup p^k \cdot \text{cl}(\text{co}Y_A^k); \tag{3}$$

$$\int_A x^k(a) d\nu(a) - \int_A e(a) d\nu(a) - y^k - z^k = 0. \tag{4}$$

$$p^k \in C^+ \cap B \;\; \text{and} \;\; \|p^k\| = 1 \text{ if } C \neq \{0\}. \tag{5}$$

The above assertions are straightforward but the last one. From Theorem 4.1, we know that we can further choose p^* such that $\|p^k\| = 1$ if $Q = C^+$ is not a subspace, or equivalently if C is not a subspace, that is, if $C \neq C \cap -C$. But this last assertion is equivalent to $C \neq \{0\}$ since the cone C is pointed.

4.4 The limit argument

The end of the proof goes as follows. The first step [Claim 4.1] shows that the sequence $\{(\int_A x^k(a) d\nu(a), y^k, p^k)\} \subset \mathbb{R}^H \times Y_A \times B$ is bounded, hence, without any loss of generality we can assume that it converges to some element $(\xi^*, y^*, p^*) \subset \mathbb{R}^H \times Y_A \times B$ and that the above Condition $(\star)$ is satisfied. The second step [Claim 4.2] shows that y^* is profit maximizing, that is, $p^* \cdot y^* = \sup p^* \cdot Y_A$. The third step applies Fatou's Lemma [Theorem B of the Appendix] to get an integrable mapping $x^* : A \to \mathbb{R}^H$ as the "limit" (in a precise sense) of a subsequence of the sequence of mapping $\{x^k\}$. The fourth step [Claim 4.4] shows that x^* satisfies the consumption equilibrium condition. The last step [Claim 4.5] shows that the market clearing condition is satisfied by (x^*, y^*). Altogether this shows that (x^*, y^*, p^*) is a weak equilibrium of the coalition production economy $\mathcal{E}_c$.

Claim 4.1 *(i) The sequence $\{(\int_A x^k(a) d\nu(a), y^k, z^k, p^k)\}$ is bounded. (ii) Hence, without any loss of generality we can assume that it converges to some element $(\xi^*, y^*, z^*, p^*) \in \mathbb{R}^H \times Y_A \times (-C) \times (C^+ \cap B)$. Furthermore if $C \neq \{0\}$, then $\|p^*\| = 1$.*

Proof. (i) We first prove that the sequence $\{(\int_A x^k(a) d\nu(a), y^k, z^k)\}$ belongs to the set

$$M := \{(\xi, y, z) \in [B(0, \overline{\rho}) + C] \times Y_A \times (-C) \mid x - y - z = \int_A e(a) d\nu(a)\}.$$

We first notice that, for every k, y^k belongs to the total production set $Y_A := \int_A Y(a) d\nu(a)$. Indeed, for every k, $Y^k(a) \subset Y(a)$, hence $Y_A^k := \int_A Y^k(a) d\nu(a) \subset \int_A Y(a) d\nu(a) = Y_A$. Consequently, $y^k \in \text{cl}(\text{co}Y_A^k) \subset Y_A$, since Y_A is closed and convex. We now show that, for every k, $\int_A x^k(a) d\nu(a) \in B(0, \overline{\rho}) + C$, where $\overline{\rho} = \int_A \rho(a) d\nu(a)$, and the function ρ is defined in Assumption $\mathbf{B}_c$. Indeed, from Assertion (2) and Assumption $\mathbf{B}_c$, we deduce that $x^k(a) \in X^k(a) \subset X(a) \subset B(0, \rho(a)) + C$. From Aumann's measurable selection theorem, there exist two measurable mappings u, v from A to $\mathbb{R}^H$ such that, for a.e. $a \in A$, $u(a) \in B(0, \rho(a)), v(a) \in C$ and $x^k(a) = u(a) + v(a)$. We notice that the mapping u is integrable (since it is integrably bounded), hence $v = x^k - u$ is also integrable. Integrating on A, we get $\int_A x^k(a) d\nu(a) = \overline{u} + \overline{v}$, with $\overline{u} \in B(0, \overline{\rho})$ and $\overline{v} \in \int_A C d\nu = C$ since C is convex. Finally, from Assertion (4) we get $\int_A x^k(a) d\nu(a) - \int_A e(a) d\nu(a) - y^k - z^k = 0$.

We end the proof of the claim by showing that the set M is bounded. It suffices to show that $\mathbf{A}M = \{0\}$ (see, for example, Debreu (1959)). One clearly has

$$\mathbf{A}M \subset \{(x, y, z) \in C \times \mathbf{A}(Y_A) \times (-C) \mid x - z = y\} = \{(0, 0, 0)\},$$

since $C \cap \mathbf{A}(Y_A) = \{0\}$, and $C \cap (-C) = \{0\}$ from Assumption $\mathbf{B}_c$.
(ii) It is a direct consequence of (i) and of Assertion (5). ∎

In the following, we let $\pi^k : A \to \mathbb{R}$, $\pi^k(a) := \sup p^k \cdot Y^k(a)$.

Claim 4.2 (i) $p^* \cdot y^* = \sup p^* \cdot Y_A$;
(ii) $p^* \cdot y^* = \liminf_k \int_A \pi^k(a) d\nu(a) = \int_A \sup p^* \cdot Y(a) d\nu(a)$;
(iii) for a.e. $a \in A$, $\sup p^* \cdot Y(a) = \text{liminf}_k \pi^k(a)$.

Proof. We first show that

$$\sup p^* \cdot Y(a) \leq \ \text{liminf}_k \pi^k(a), \text{ for a.e. } a \in A. \tag{6}$$

Indeed, let $y \in Y(a)$, then, for k large enough, $y \in Y^k(a)$, hence $p^k \cdot y \leq \pi^k(a)$. Taking the liminf, when $k \to \infty$, we get $p^* \cdot y = \text{liminf}_k p^k \cdot y \leq \text{liminf}_k \pi^k(a)$. Taking the supremum over $Y(a)$, we get (6).

From (6) and the fact that $\sup p^k \cdot Y_A^k = \sup p^k \cdot \text{cl}(\text{co}Y_A^k) = p^k \cdot y^k$ (by Assertion (3)) we get

$$\begin{aligned}
\sup p^* \cdot Y_A &= \sup p^* \cdot \int_A Y(a) d\nu(a) \\
&= \int_A \sup p^* \cdot Y(a) d\nu(a) \leq \int_A \ \text{liminf}_k \pi^k(a) d\nu(a) \\
&\leq \ \text{liminf}_k \int_A \pi^k(a) d\nu(a) \\
&= \ \text{liminf}_k \sup p^k \cdot \int_A Y^k(a) d\nu(a) = \ \text{liminf}_k \sup p^k \cdot Y_A^k \\
&= \ \text{liminf}_k p^k \cdot y^k = p^* \cdot y^* \leq \sup p^* \cdot Y_A.
\end{aligned}$$

This ends the proof of Part (i) and Part (ii) of the claim.

Finally, Part (iii) is a consequence of (6) and the fact that we have shown above that: $\int_A [\sup p^* \cdot Y(a) - \ \text{liminf}_k \pi^k(a)] d\nu(a) = 0$. ∎

Claim 4.3 *There exists an integrable mapping $x^* : A \to \mathbb{R}^H$ such that:*

$$\int_A x^*(a) d\nu(a) \leq_C \lim_k \int_A x^k(a) d\nu(a) = \xi^*$$

for a.e. $a \in A$, $(x^*(a), \ \sup p^* \cdot Y(a))$ is adherent to the sequence $\{(x^k(a), \pi^k(a))\}$.

Proof. We will apply Fatou's lemma [Theorem B of the Appendix] to the sequence of mappings $\{(x^k, \pi^k)\}$ and to the cone $C \times \mathbb{R}_+$. We know that the sequence $\{x^k\}$ is C-integrably bounded (from Assumption $\mathbf{B}_c$) and that $\lim_k \int_A x^k(a) d\nu(a) = \xi^*$ (from Claim 4.1). We now show that the sequence of integrable functions $\pi^k : A \to \mathbb{R}$ is integrably bounded below. Indeed, from Lemma 4.1 and the definition of $Y^k(a)$, for a.e. $a \in A$, $\pi^k(a) \geq p^k \cdot \hat{y}(a)$. Hence, using the fact that the sequence $\{p^k\}$ is convergent, we deduce that $\{\pi^k\}$ is integrably bounded below. Finally, from Claim 4.2, without any loss of generality, we can assume that $\lim_k \int_A \pi^k(a) d\nu(a) = p^* \cdot y^*$.

All the assumptions of Fatou's Lemma [Theorem B in Appendix] are satisfied by the sequence $\{(x^k, \pi^k)\}$ and the cone $C \times \mathbb{R}_+$. Consequently there exist integrable mappings $x^* : A \to \mathbb{R}^H$ and $\pi^* : A \to \mathbb{R}$ satisfying the following three conditions:

$$z^{**} := \int_A x^*(a)d\,\nu(a) - \lim_k \int_A x^k(a)d\nu(a) \leq_C 0; \tag{7}$$

$$\int_A \pi^*(a)d\nu(a) - \lim_k \int_A \pi^k(a)d\nu(a) \leq 0; \tag{8}$$

$$\text{for a.e. } a \in A, (x^*(a), \pi^*(a)) \text{ is adherent to the sequence } \{(x^k(a), \pi^k(a))\}. \tag{9}$$

To end the proof of the claim, we only have to show that $\pi^*(a) = \sup p^* \cdot Y(a)$, for a.e. $a \in A$. Indeed, since from (9), for a.e. $a \in A$, $\pi^*(a)$ is adherent to the sequence $\{\pi^k(a)\}$, using Claim 4.2, we deduce that

$$\sup p^* \cdot Y(a) \leq \liminf_k \pi^k(a) \leq \pi^*(a), \text{ for a.e. } a \in A. \tag{10}$$

Consequently, using (8) and Claim 4.2, we get

$$\begin{aligned} \int_A \sup p^* \cdot Y(a)d\nu(a) \quad &\leq \quad \int_A \pi^*(a)d\nu(a) \leq \lim_k \int_A \pi^k(a)d\nu(a) \\ &= \quad p^* \cdot y^* = \int_A \sup p^* \cdot Y(a)d\nu(a) \end{aligned}$$

This implies that $\int_A[\pi^*(a) - \sup p^* \cdot Y(a)]d\nu(a) = 0$, which, together with (10), proves that $\pi^*(a) = \sup p^* \cdot Y(a)$ for a.e. $a \in A$. This ends the proof of the claim. ∎

Before stating the next claim, we let

$$r^*(a) := p^* \cdot e(a) + \sup p^* \cdot Y(a) + \delta(a)(1 - \|p^*\|),$$

and we recall that

$$r^k(a, p^k) \quad := \quad p^k \cdot e(a) + \pi^k(a) + \delta(a)(1 - \|p^k\|).$$

From Claim 4.3 and the fact that $\{p^k\}$ converges to p^*, we deduce that

$$\text{for a.e. } a \in A,\ (x^*(a), r^*(a)) \text{ is adherent to the sequence } \{(x^k(a), r^k(a, p^k))\}. \tag{11}$$

Claim 4.4 *For a.e. $a \in A$*
(i) $x^*(a) \in B(a, p^*, r^*(a)) := \{x \in X(a) \mid p^* \cdot \leq r^*(a)\}$;
(ii) for every $x \in X(a)$, $x^*(a) \prec_a x$ implies $r^*(a) \leq p^* \cdot x$.

Proof. (*i*) From (11), $x^*(a)$ is adherent to the sequence $\{x^k(a)\}$ for a.e. $a \in A$. Hence $x^*(a) \in X(a)$, since $X(a)$ is closed (by Assumption **C**). Furthermore, from Assertion (2), $x^k(a) \in D^k(a, p^k, r^k(a, p^k))$, hence $p^k \cdot x^k(a) \leq r^k(a, p^k)$, which implies, using Assertion (11), that $p^* \cdot x^*(a) \leq r^*(a)$. Hence $x^*(a) \in B(a, p^*, r^*(a))$.
(*ii*) By contraposition. Suppose that there exists $x \in X(a)$ such that $x^*(a) \prec_a x$ and $p^* \cdot x < r^*(a)$. From (11), passing to a subsequence if it is necessary, we can suppose that $\{x^k(a)\}$ converges to $x^*(a)$ and $\{r^k(a, p^k)\}$ converges to $r^*(a)$. Since $x \in X(a)$, for k large enough, one has $x \in X^k(a)$. From the continuity of $\prec_a$ (by Assumption in **C**), the set $\{x' \in X(a) \mid x' \prec_a x\}$ is open in $X(a)$ (for its relative topology) and contains $x^*(a)$, hence, for k large enough, $x^k(a) \prec_a x$. But this, together with the fact that $x^k(a) \in D^k(a, p^k, r^k(a, p^k))$ (by Assertion (2)) implies that $p^k \cdot x \geq r^k(a, p^k)$ and this contradicts the fact that, for k large enough, $p^k \cdot x < r^k(a, p^k)$ (since $p^* \cdot x < r^*(a)$). This ends the proof of the claim. ∎

Claim 4.5 $\int_A x^*(a)d\nu(a) - \int_A e(a)d\nu(a) - y^* \leq_C 0$.

Proof. From Assertion (7) and Assertion (4) we deduce that

$$\int_A x^*(a)d\nu(a) = \int_A e(a)d\nu(a) + y^* + z^* + z^{**} \leq_C \int_A e(a)d\nu(a) + y^*,$$

which ends the proof of the claim. ■

Remark 4.1 When A is finite, the above proof allows to show the existence of a quasi-equilibrium with slack when $C \neq \{0\}$, with only some slight changes. When A is not finite however, our proof does not carry when $C \neq \{0\}$ and this is due to the use of Fatou's lemma. This last argument is difficult to bypass, and a different method of proof would certainly be needed to consider the case $C \neq \{0\}$, if any. The same difficulty arises with the concepts of monetary equilibria of Kajii (1996) and weak equilibria of Polemarchakis and Siconolfi (1993). Even if free disposal equilibria are less satisfactory than the previous concepts for welfare reasons (see the example in Section 2.4), among the various equilibrium concepts existing in the literature in such a framework, this is the only one for which the existence can be proved when $C \neq \{0\}$, hence also in the important case $C = \mathbb{R}^H_+$.

5 Appendix : A Version of Fatou's lemma

This version of Fatou's lemma is used in the proof of Theorem 3.1 [in Section 4].

Theorem B *Let C be a pointed, closed, convex cone in $\mathbb{R}^H$, and let $\{f^k\}$ be a sequence of integrable mappings, from a complete finite positive measure space $(A, \mathcal{A}, \nu)$ to $\mathbb{R}^H$, such that (i) $\lim_k \int_A f^k(a)d\nu(a)$ exists, and (ii) there is an integrable mapping $g : A \to \mathbb{R}_+$ such that, for all k, for a.e. a, $f^k(a) \in B(0, g(a)) + C$.*

Then there exists an integrable mapping $f : A \to \mathbb{R}^H$ such that:

- *for a.e. $a \in A$, $f(a)$ is adherent to the sequence $\{f^k(a)\}$;*
- *$\int_A f(a)d\nu(a) \leq_C \lim_k \int_A f^k(a)d\nu(a)$.*

Theorem B generalizes and presents in a synthetic way two standard versions of Fatou's lemma which are usually considered in the same type of models. The first one considers the case $C = \{0\}$ and is due to Arstein (1979) (under the weaker assumption that the sequence $\{f^k\}$ is uniformly integrable). The second one considers the case $C = \mathbb{R}^H_+$ and is due to Schmeidler (1970) (in which Condition (ii) is equivalent to the existence of an integrable mapping $\underline{f} : A \to \mathbb{R}^H$ such that, for all k and for a.e. $a \in A$, $f^k(a) \geq \underline{f}(a)$).

The proof is a consequence of the following result due to Balder and Hess (1995).

Theorem C *Let $\{f^k\}$ be a sequence of integrable mappings, from a complete finite positive measure space $(A, \mathcal{A}, \nu)$ to $\mathbb{R}^H$, such that (i) $\lim_k \int_A f^k(a)d\nu(a)$ exists and (ii') $sup_k \int_A \|f^k(a)\|d\nu(a) < \infty$. Then there exists an integrable mapping $f : A \to \mathbb{R}^H$ such that*

- *for a.e. $a \in A$, $f(a)$ is adherent to $\{f^k(a)\}$,*
- *$\lim_k \int_A f^k(a)d\nu(a) \in \int_A f(a)d\nu(a) + (C')^+$, where*

$$C' := \{x \in \mathbb{R}^H \mid \text{ the sequence } \{max\{0, -< x, f^k(.) >\}\}_k \text{ is uniformly integrable}\}.$$

We now give the proof of Theorem B, and prepare it with a claim.

Claim 5.1 *Under the assumptions of Theorem B,* $sup_k \int_A \|f^k(a)\| d\nu(a) < \infty$.

Proof. Let $e \in \mathbb{R}^H$ such that $e \in \text{ int}C^+$, which exists since C is pointed (see Rockafellar (1970)). We claim that there exists $m > 0$, such that

$$\|x\| \leq m < e, x > \text{ for all } x \in C. \tag{12}$$

Indeed, we define the set $S_e := \{x \in C \mid < e, x >= 1\}$, which is nonempty and compact (see Rockafellar (1970)). We let $m = \max \{\|x\| \mid x \in S_e\}$, and clearly $m > 0$. But for $x \in C \setminus \{0\}$, recalling that $e \in \text{ int}C^+$, one has $< e, x >> 0$ (see Rockafellar (1970)). Consequently, for every $x \in C$ one has $\|x/ < e, x > \| \leq m$, which clearly implies (12).

From Assumption (ii) of Theorem B, there exist measurable mappings $\underline{f}^k : A \to \mathbb{R}^H$ such that, for all k and a.e. $a \in A$, one has $f^k(a) - \underline{f}^k(a) \in C$ and $\underline{f}^k(a) \in B(0, g(a))$. This is a consequence of Aumann's measurable selection theorem, applied to the measurable correspondence $a \to \{(u, v) \in B(0, g(a)) \times C \mid f_k(a) = u + v\}$. We can further assume the mappings $\underline{f}^k$ to be integrable, since the sequence $\{\underline{f}^k\}$ is integrably bounded. From above, using Assertion (12), we get

$$\begin{aligned}
\int_A \|f^k(a)\| d\nu(a) &\leq \int_A \|f^k(a) - \underline{f}^k(a)\| d\nu(a) + \int_A \|\underline{f}^k(a)\| d\nu(a) \\
&\leq m \int_A < e, f^k(a) - \underline{f}^k(a) > d\nu(a) + \int_A g(a) d\nu(a) \\
&= m < e, \int_A [f^k(a) - \underline{f}^k(a)] d\nu(a) > + \int_A g(a) d\nu(a) \\
&\leq m\|e\| \| \int_A f^k(a) d\nu(a) - \int_A \underline{f}^k(a) d\nu(a)\| + \int_A g(a) d\nu(a) \\
&\leq m\|e\| \| \int_A f^k(a) d\nu(a)\| + (m\|e\| + 1) \int_A g(a) d\nu(a).
\end{aligned}$$

Consequently, recalling that the sequence $\{\int_A f^k(a) d\nu(a)\}$ is convergent, one gets $\sup_k \int_A \|f^k(a)\| d\nu(a) \leq m\|e\| \sup_k \| \int_A f^k(a) d\nu(a)\| + (m\|e\| + 1) \int_A g(a) d\nu(a) < \infty$. ■

Proof of Theorem B. From Claim 5.1, the sequence $\{f^k\}$ satisfies the assumptions of Balder's Theorem. Consequently, there exists an integrable mapping $f : A \to \mathbb{R}^H$, such that for a.e. $a \in A$, $f(a)$ is adherent to $\{f^k(a)\}$ and

$$\lim_k \int_A f^k(a) d\nu(a) \in \int_A f(a) d\nu(a) + (C')^+, \tag{13}$$

where C' is the set of elements $x \in \mathbb{R}^H$ such that the sequence of real-valued functions $\{\max\{0, - < x, f^k(.) >\}\}_k$ is uniformly integrable.

In view of Assertion (13), the proof of Theorem B will be complete if we show that $C'^+ \subset C$. But this inclusion will hold if we show that $C^+ \subset C'$, since it implies that $C'^+ \subset C^{++} = C$, from the bipolar theorem (see Rockafellar (1970)), recalling that C is a closed convex cone. We now prove that $C^+ \subset C'$. Indeed, let $x \in C^+$, from Assumption (ii) of Theorem B, for a.e. $a \in A$

$$\begin{aligned}
0 &\leq \max\{0, - < x, f^k(a) >\} \\
&\leq \max\{\{0, < x, u + v >\} | u \in -B(0, g(a)), v \in -C\} \leq \|x\| g(a),
\end{aligned}$$

which implies that the sequence of real-valued functions $\{\max \{0, - < x, f^k(.) >\}\}_k$ is integrably bounded, hence is uniformly integrable. Consequently, $C^+ \subset C'$ and this ends the proof of Theorem B. ■

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