Euler Characteristic and Fixed-Point Theorems

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Abstract. We propose a geometric definition of the Euler characteristic $\chi(M)$ for the class of compact epi-Lipschitzian sets $M \subset \mathbb{R}^n$ and we provide existence theorems of (generalized) equilibria for set-valued mappings F when the domain M of F is neither assumed to be convex, nor smooth but has a nonzero Euler characteristic.

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1. Introduction

We propose a geometric definition of the Euler characteristic $\chi(M)$ for the class of compact epi-Lipschitzian sets $M \subset \mathbb{R}^n$ and we provide existence theorems of (generalized) equilibria for set-valued mappings F when the domain M of F is neither assumed to be convex, nor smooth but has a nonzero Euler characteristic.

When $M \subset \mathbb{R}^n$ is smooth, i.e., a closed C^1 submanifold with boundary of $\mathbb{R}^n$, the Euler characteristic of M can be defined as follows [see Milnor [29]]. For every $x \in \partial M$, the boundary of M, we let $G_M(x)$ be the unique outward normal vector to M at x, in the unit sphere S of $\mathbb{R}^n$. Then the mapping $G_M : \partial M \to S$, called the Gauss mapping, is continuous and everywhere different from 0. The Euler characteristic of M, denoted by $\chi(M)$, is then defined to be the topological degree of the mapping G_M with respect to the point 0, that is, $\chi(M) := \deg(G_M, 0)$.

In the absence of smoothness assumptions on the set M, we shall follow the same approach, with the only difference that the Gauss mapping will be multivalued, and the topological degree will thus need to be defined for set-valued mappings. In this paper, we consider the class of closed epi-Lipschitzian subsets M of $\mathbb{R}^n$, that is, for every x in M, Clarke's normal cone to the set M at x, denoted $N_M(x)$, is assumed to be pointed, i.e., $N_M(x) \cap -N_M(x) = \{0\}$. We point out that the class of closed epi-Lipschitzian sets contains, first, C^1 submanifolds with boundaries of $\mathbb{R}^n$ and, second, closed convex subsets of $\mathbb{R}^n$ with a nonempty interior. It contains also sets defined by finitely many continuously differentiable inequality constraints, which satisfy a Constraint Qualification Assumption. The Gauss set-valued mapping G_M of an epi-Lipschitzian set $M \subset \mathbb{R}^n$, is then defined, for $x \in \partial M$, by $G_M(x) = \mathrm{co}\,[N_M(x) \cap S]$ (where "co" denotes the convex hull), a definition, which generalizes the one presented before. As previously, we let $\chi(M) := \deg\,(G_M, 0)$, that is, the degree of the Gauss set-valued mapping G_M, which is shown to be upper semicontinuous, with nonempty, convex, compact values that do not contain the point 0.

The main aim of this paper is then to prove existence theorems of equilibria for correspondences F, from M to $\mathbb{R}^n$, satisfying a tangential condition, or existence theorems of generalized equilibria when the correspondence F satisfies no tangential condition. The domain M of F is neither supposed to be convex, nor smooth but is assumed to be compact, epi-Lipchitzian, with a nonzero Euler characteristic (this last assumption being always satisfied in the convex case). This allows to provide a unified approach that contains both Brouwer–Kakutani type fixed-point theorems when M is convex, and Poincaré–Hopf's theorem when M is smooth. The approach to the existence problem that we consider here uses degree theory and gives also some information on the structure of the equilibrium set (or the the fixed-point set), allowing, for example, counting the number of equilibria, modulo 2.

The paper is organized as follows. The first section recalls the definition and the main properties of the topological degree of a set-valued mapping, together with the notions of Clarke's tangent and normal cones. General references on these two subjects are [8, 26], and [11], respectively. Section 3 is devoted, first, to the definitions and properties of the Gauss set-valued mapping G_M for the class of closed epi-Lipschitzian sets $M \subset \mathbb{R}^n$. Second, we define the Euler characteristic $\chi(M)$ for this class of sets and prove several properties, including a characterization, using the dual notion of tangent cones. In Section 4, we provide existence theorems of equilibria (resp. generalized equilibria) for correspondences F, from M to $\mathbb{R}^n$, satisfying a tangential condition (resp. when F does not satisfy any tangential condition) and M is assumed to be compact epi-Lipschitzian with a nonzero Euler characteristic. Finally, in Section 5, we recall that every nonempty compact epi-Lipschitzian set $M \subset \mathbb{R}^n$ can be defined by a nondegenerate Lipschitz inequality, that is, $M = \{x \in \mathbb{R}^n \mid \varphi(x) \leq 0\}$, where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz and its generalized gradient $\partial\varphi(x)$ does not contain 0 whenever $\varphi(x) = 0$. This allows to give additional ways to compute the Euler characteristic $\chi(M)$, in terms of the degree of the set-valued mapping $x \to \partial\varphi(x)$.

2. Preliminaries

A[1] set-valued mapping F, from $M \subset \mathbb{R}^n$ to $\mathbb{R}^p$, is a mapping from M to the set of all subsets of $\mathbb{R}^p$. It is said to be upper semicontinuous (or, simply, u.s.c.) if, for every open subset V of $\mathbb{R}^p$, the following set $\{x \in M \mid F(x) \subset V\}$ is open in M (for its relative topology).

2.1. Topological Degree of a Set-valued Mapping

Let D be an open bounded subset of $\mathbb{R}^n$, and let $y \in \mathbb{R}^n$, we denote by $\mathcal{C}(\mathrm{cl}\, D, \mathbb{R}^n; y)$ or, simply by $\mathcal{C}$ when no confusion is possible, the set of set-valued mappings F, from $\mathrm{cl}\, D$ to $\mathbb{R}^n$, satisfying the two following properties: (i) F is upper semicontinous, with nonempty, convex, and compact values, and (ii) $y \notin F(x)$ if $x \in \partial D$.

For every $F \in \mathcal{C}(\mathrm{cl}\, D, \mathbb{R}^n; y)$, we denote by $\deg(F, y)$ the topological degree of F, which we recall is an integer. General references on this subject are J. T. Schwartz [35], or Ortega and Rheinboldt [30] when F is single-valued (hence reduces to a continuous mapping), and Granas [26] and Cellina and Lasota [8] for the extended definition to set-valued mappings. We recall hereafter some properties of the degree that will be used in the following.

Property D.1 *(Invariance Under Homotopy Avoiding y)* Let H be an upper semicontinuous set-valued mapping from $[0,1] \times \mathrm{cl}\, D$ to $\mathbb{R}^n$, with nonempty, convex, compact values, which avoids y, in the sense that, for every $(t, x) \in [0,1] \times \partial D$, $y \notin H(t, x)$. Then

$$\deg(H_0, y) = \deg(H_1, y), \text{ where } H_0 = H(0, \cdot), H_1 = H(1, \cdot).$$

Property D.2 *(Existence Property)* For every F in $\mathcal{C}(\mathrm{cl}\, D, \mathbb{R}^n; y)$ such that $\deg(F, y) \neq 0$, then there exists $x^* \in \mathrm{cl}\, D$ such that

$$y \in F(x^*).$$

[1] If $x = (x_i)$ and $y = (y_i)$ are two vectors of $\mathbb{R}^n$, we denote by $x \cdot y = \sum_{i=1}^n x_i y_i$, the scalar product of $\mathbb{R}^n$, by $\|x\| = (x \cdot x)^{1/2}$ the Euclidean norm. The closed Euclidean balls and spheres are denoted by $B(x, r) = \{y \in \mathbb{R}^n | \|y - x\| \leq r\}$ and $S(x, r) = \{y \in \mathbb{R}^n | \|y - x\| = r\}$, respectively, and $B = B(0, 1)$, $S = S(0, 1)$. If A and B are two subsets of $\mathbb{R}^n$, we denote by $A \setminus B := \{x \in A \mid x \notin B\}$, the set difference of the two sets and by $A + B = \{a + b \mid a \in A, b \in B\}$, the sum of the two sets ; we denote by $\mathrm{cl}\, A$, the closure of A, $\mathrm{int} A$, the interior of A, ∂A, the boundary of A, and $A^o = \{p \in \mathbb{R}^n \mid p \cdot a \leq 0 \text{ for every } a \in A\}$, the polar cone of A; for $x \in \mathbb{R}^n$, we also let $d_A(x) = \inf\{\|x - a\| \mid a \in A\}$, the distance from x to A.

Property D.3 *(Degree Formula)* For every single-valued[2] mapping F in $\mathcal{C}(\mathrm{cl}\, D, \mathbb{R}^n; y)$ such that the restriction of F on D is continuously differentiable, and y is a regular value[3] of F, then

$$\deg(F, y) = \sum_{x \in F^{-1}(y)} \mathrm{sg}\ \det\ F'(x).$$

Property D.4 *(Excision Property)* Let $K \subset \mathbb{R}^n$ be a compact, let F be a set-valued mapping in $\mathcal{C}(\mathrm{cl}\, D, \mathbb{R}^n; y)$ such that $y \notin F(x)$ for every $x \in K$, we denote by $F_{|\mathrm{cl}[D \setminus K]}$ the restriction of F to $\mathrm{cl}[D \setminus K]$. Then

$$\deg(F, y) = \deg(F_{|\mathrm{cl}[D \setminus K]}, y).$$

2.2. Some further properties of the degree

We give some additional properties of the degree that will be used hereafter.

Property D.5 Let F_0, F_1 belong to $\mathcal{C}(\mathrm{cl}\, D, \mathbb{R}^n; y)$ and satisfy

$$y \notin \ \mathrm{co}\ [F_0(x) \cup F_1(x)] \text{ for every } x \in \partial D,$$

then $\deg(F_0, y) = \deg(F_1, y)$.

Proof. It is a consequence of the Homotopy Property D.1. Indeed, consider the set-valued mapping H, from $[0,1] \times \mathrm{cl}\, D$ to $\mathbb{R}^n$, defined by $H(t, x) = (1-t)F_0(x) + tF_1(x)$. Then H is clearly u.s.c. with nonempty, convex, compact values. Moreover H avoids the point y. Indeed, if it is not true, there exists some $(t, x) \in [0,1] \times \partial D$ such that $y \in H(t, x) \subset \ \mathrm{co}\ [F_0(x) \cup F_1(x)]$, a contradiction. From Property D.1, we deduce that $\deg(F_0, y) = \deg(H_0, y) = \deg(H_1, y) = \deg(F_1, y)$. □

We now give a direct consequence of Property D.5.

Property D.6 *(Dependence Only on the Boundary Values)* If F_0 and F_1 belong to $\mathcal{C}(\mathrm{cl}\, D, \mathbb{R}^n; y)$ and satisfy

$$F_0(x) \subset F_1(x) \ \text{ for every } x \in \partial D,$$

(hence in particular if F_0 and F_1 coincide on ∂D), then

$$\deg(F_0, y) = \deg(F_1, y).$$

[2] That is, there exists a mapping $f : \mathrm{cl}\, D \to \mathbb{R}^n$ such that $F(x) = \{f(x)\}$ and we notice that f is continuous if F is u.s.c.. When there is no risk of confusion, the mapping f will also be denoted by F.

[3] That is, for every $x \in F^{-1}(y) := \{x \in D | F(x) = y\}$, then $\det F'(x) \neq 0$, where $F'(x)$ denotes the derivative of F at x and $\det F'(x)$ the determinant of $F'(x)$. We also let $\mathrm{sg}\ \det F'(x) = +1$ (resp. -1) if $\det F'(x) > 0$ (resp. < 0).

We now define the degree of a set-valued mapping defined on ∂D [instead of $\mathrm{cl}\, D$, as previously]. We denote by $\mathcal{C}(\partial D, \mathbb{R}^n; y)$ the set of u.s.c. set-valued mappings, from ∂D to $\mathbb{R}^n$, such that, for every $x \in \partial D$, $F(x)$ is nonempty, convex, compact, and $y \notin F(x)$.

DEFINITION 2.1. *Let $F \in \mathcal{C}(\partial D, \mathbb{R}^n; y)$, then the topological degree of F, also denoted by* $\deg(F, y)$, *is defined by:*

$$\deg(F, y) := \deg(\hat{F}, y),$$

where $\hat{F}$ is an arbitrary set-valued mapping in $\mathcal{C}(\mathrm{cl}\, D, \mathbb{R}^n; y)$, which extends F to $\mathrm{cl}\, D$, i.e., $F(x) = \hat{F}(x)$ for every $x \in \partial D$.

The above definition has a clear meaning since, first, such an extension always exists [see, for example Cellina [9]] and, second, the definition is independent of the choice of the extension, by Property D.6.

We end this section with a last property.

Property D.7 Let F_0, F_1 belong to $\mathcal{C}(\mathrm{cl}\, D, \mathbb{R}^n; y)$ and satisfy

$$F_0(x) \cap F_1(x) \neq \emptyset \text{ for every } x \in \partial M,$$

then $\deg(F_0, y) = \deg(F_1, y) = \deg(F_0 \cap F_1, y)$.

Proof. From Property D.6, we have $\deg(F_0 \cap F_1, y) = \deg(F_0, y)$, and $\deg(F_0 \cap F_1, y) = \deg(F_1, y)$, hence the result. □

2.3. Clarke's normal and tangent cones

Let M be a closed subset of $\mathbb{R}^n$, and let $x \in M$, we define Clarke's normal (resp. tangent) cone $N_M(x)$ (resp. $T_M(x)$) to M at x as follows. We first define the set of perpendicular vectors to M at $x \in M$ (also called proximal normal vectors) to be the set:

$$\perp_M (x) := \{p \in \mathbb{R}^n \mid \exists \varepsilon > 0 : \mathrm{int}\, B(x + \varepsilon p, \varepsilon \|p\|) \cap M = \emptyset\}.$$

Equivalently, the vector p is perpendicular to M at x if for $\varepsilon > 0$ sufficiently small, x is a nearest point in M to $x + \varepsilon p$ (for the Euclidean norm).

Then Clarke's normal cone to M at x, denoted by $N_M(x)$, is the closed convex hull of the set:

$$\{p \in \mathbb{R}^n \mid \exists (x^\nu, p^\nu) \subset M \times \mathbb{R}^n, (x^\nu, p^\nu) \to (x, p), \forall \nu : \ p^\nu \in \perp_M (x^\nu)\}.$$

We then define Clarke's tangent cone by polarity as follows:

$$T_M(x) = N_M(x)^o := \{v \in \mathbb{R}^n \mid p \cdot v \leq 0 \text{ for every } \quad p \in N_M(x)\}.$$

We now recall some properties of the tangent and normal cones that will be used hereafter. A general reference on this subject is Clarke [11].

PROPOSITION 2.1. *Let M be a closed subset of $\mathbb{R}^n$ and let $x \in M$.*
(a) Then $N_M(x)$ and $T_M(x)$ are closed convex cones containing 0 *and*

$$N_M(x) \cap T_M(x) = \{0\}.$$

(b) The following assertions are equivalent
(i) $x \in \partial M$, (ii) $N_M(x) \neq \{0\}$, and (iii) $T_M(x) \neq \mathbb{R}^n$.
(c) Suppose that M is a C^1 submanifold with boundary of $\mathbb{R}^n$. Then, for every $x \in \partial M$, Clarke's normal cone $N_M(x)$ is the closed half-line spanned by the unit outward normal vector $G_M(x)$ to M at x, that is,

$$N_M(x) = \{\lambda G_M(x) \mid \lambda \geq 0\}.$$

(d) Suppose that M is convex, then Clarke's normal and tangent cones coincide, respectively, with the cone of normal and tangent vectors in the sense of convex analysis, that is, for every $x \in M$

$$N_M(x) = \{p \in \mathbb{R}^n \mid p \cdot x \geq p \cdot x' \text{ for every } x' \in M\},$$

$$T_M(x) = \mathrm{cl}\,\{\lambda(x' - x) \mid \lambda > 0, x' \in M\}.$$

3. The Euler Characteristic of an Epi-Lipschitzian Set

3.1. The Gauss Mapping of an Epi-Lipschitzian set

When M is a compact C^1 submanifold with boundary of $\mathbb{R}^n$, the Euler characteristic of M can be defined as follows [see Milnor [29]]. For every $x \in \partial M$, the boundary of M, we let $G_M(x)$ be the unit outward normal vector to M at x. Then the mapping $G_M : \partial M \to S$, called the Gauss mapping, is continuous and is everywhere different from 0. The Euler characteristic of M, denoted by $\chi(M)$, is then defined to be the degree of G_M with respect to the point $y = 0$, that is

$$\chi(M) := \deg(G_M, 0).$$

In the absence of differentiability assumptions on the set M, we shall follow the same approach with the only difference that the Gauss mapping will be multivalued. We now define the class of sets which will be considered throughout this paper.

DEFINITION 3.1 (Rockafellar [33]). *A closed subset M of $\mathbb{R}^n$ is said to be epi-Lipschitzian if, for every $x \in M$, the normal cone $N_M(x)$ is pointed, i.e., $N_M(x) \cap -N_M(x) = \{0\}$, or equivalently, if $T_M(x)$ has a nonempty interior.*

We point out that the class of epi-Lipschitzian sets contains, first, C^1 submanifolds with boundaries of $\mathbb{R}^n$ and, second, closed convex subsets of $\mathbb{R}^n$ with a nonempty interior [Proposition 2.1 (c) and (d), respectively]. It contains also sets defined by continuously differentiable inequality constraints which satisfy a Constraint Qualification Assumption [see Section 5].

We now can define the Gauss set-valued mapping for epi-Lipschitzian subsets of $\mathbb{R}^n$, as follows.

DEFINITION 3.2. *Let M be a closed epi-Lipschitzian subset of $\mathbb{R}^n$, the set-valued mapping G_M, from ∂M to $\mathbb{R}^n$, defined by*

$$G_M(x) = \mathrm{co}\,[N_M(x) \cap S]$$

is called the Gauss set-valued mapping of M, or simply the Gauss mapping of M.

The above definition generalizes the classical one presented before. Indeed, if M is a C^1 submanifold with boundary of $\mathbb{R}^n$, then, for every $x \in \partial M$, $G_M(x)$ reduces to the unit outward normal vector to M at x, by Proposition 2.1.c.

We now give some properties of the Gauss set-valued mapping G_M.

PROPOSITION 3.1. *Let M be a closed epi-Lipschitzian subset of $\mathbb{R}^n$.*
(a) [Rockafellar [33]] Then $M = \mathrm{cl}\,\mathrm{int} M$, hence $\partial M = \partial\,\mathrm{int}\, M$. Furthermore, the set-valued mapping $x \to N_M(x) \cap S$ is upper semicontinuous, with nonempty, compact values and the set-valued mappings $x \to T_M(x)$ and $x \to \mathrm{int}\, T_M(x)$ are lower semicontinuous, with nonempty, convex values.
(b) The Gauss set-valued mapping G_M, from ∂M to $\mathbb{R}^n$, is upper semicontinuous with nonempty, convex, compact values and, for every $x \in \partial M$, $0 \notin G_M(x)$. Furthermore, there exists an upper semicontinuous set valued-mapping $\hat{G}_M$, from M to $\mathbb{R}^n$, with nonempty, convex, compact values, such that $\hat{G}_M(x) = G_M(x)$ for every $x \in \partial M$.

Proof of Part (b). The Gauss mapping G_M is clearly upper semicontinuous since it is the convex hull of an upper semicontinuous set-valued mapping with nonempty, compact values, by Part (a).

We now show by contraposition that, for every $x \in \partial M$, $0 \notin G_M(x)$. Indeed, suppose that, for some $x \in \partial M$, $0 = \sum_{i=1}^{k} \lambda_i p_i$, for some $p_i \in N_M(x) \cap S$ and some $\lambda_i > 0$ $(i = 1, \ldots, k)$ such that $\sum_{i=1}^{k} \lambda_i = 1$. We notice that $k > 1$ since $p_1 \neq 0$. Consequently

$$p_1 = -\frac{1}{\lambda_1} \sum_{i=2}^{k} \lambda_i p_i \in S \cap N_M(x) \cap -N_M(x),$$

which contradicts the fact that $N_M(x)$ is pointed. The existence of the extension $\hat{G}_M$ is a consequence of Cellina [9]. □

3.2. The Euler characteristic of an Epi-Lipschitzian set

In view of Proposition 3.1 and Definition 2.1, we can now introduce the following definition which has a clear meaning.

DEFINITION 3.3. *Let M be a compact, epi-Lipschitzian subset of $\mathbb{R}^n$, the Euler characteristic of M, denoted by $\chi(M)$, is the integer:*

$$\chi(M) := \deg(G_M, 0),$$

where G_M is the Gauss set-valued mapping of M, and

$$\chi(M) := \deg(\hat{G}_M, 0),$$

for every set-valued mapping $\hat{G}_M$, as in Proposition 3.2.b, called an extended Gauss mapping of M, in the following.

The following propositions give different ways to compute the Euler characteristic of the set M.

PROPOSITION 3.2. *Let $M \subset \mathbb{R}^n$ be compact and epi-Lipschitzian, and let G be an u.s.c. set-valued mapping, from ∂M to $\mathbb{R}^n$, with nonempty, convex, compact values such that*

$$G(x) \cap -N_M(x) = \emptyset \quad \text{for every} \quad x \in \partial M.$$

Then, $\chi(M) = \deg(G, 0) = \deg(\hat{G}, 0)$, for every u.s.c. set-valued mapping $\hat{G}$, from M to $\mathbb{R}^n$, with nonempty, convex, compact values, which extends G to M.

Proof. Let $\hat{G}_M$ be an extended Gauss mapping of M. Recalling that $\chi(M) = \deg(G_M, 0) = \deg(\hat{G}_M, 0)$, by Definition 3.3, and that $\deg(G, 0) = \deg(\hat{G}, 0)$, by Definition 2.1, it suffices to show that

$\deg(\hat{G}_M, 0) = \deg(\hat{G}, 0)$. This is clearly a consequence of Property D.5 of the degree and of the following claim:

$$0 \notin \text{ co } [G_M(x) \cup G(x)] \text{ for every } x \in \partial M.$$

We prove the claim by contraposition. If it is not true, then one has $0 = ty + (1-t)z$, for some $(t, x, y, z) \in [0,1] \times \partial M \times G(x) \times G_M(x)$. One notice that $t \neq 0$ since $0 \notin G_M(x)$. Thus $y = -\frac{1-t}{t} z \in G(x) \cap -N_M(x)$, a contradiction with the assumption made in the proposition. $\square$

We now give a direct consequence of the previous proposition.

PROPOSITION 3.3. *Let M be a compact, epi-Lipschitzian subset of $\mathbb{R}^n$ and let G be an u.s.c. set-valued mapping, from ∂M to $\mathbb{R}^n$, with nonempty, convex, compact values such that*

$$0 \notin G(x) \text{ and } G(x) \cap N_M(x) \neq \emptyset \quad \text{for every} \quad x \in \partial M.$$

Then, $\chi(M) = \deg(G, 0) = \deg(\hat{G}, 0)$, for every u.s.c. set-valued mapping $\hat{G}$, from M to $\mathbb{R}^n$, with nonempty, convex, compact values, which extends G to M.

Proof. We define the set-valued mapping G_0, from ∂M to $\mathbb{R}^n$, as follows $G_0(x) = G(x) \cap N_M(x)$. From Proposition 3.1.a, we deduce that G_0 is an u.s.c. set-valued mapping, with nonempty, convex, compact values. Consequently, from Property D.6 of the degree we get

$$\deg(G_0, 0) = \deg(G, 0).$$

In view of the above equality, the proof will be complete if we show that $\chi(M) = \deg(G_0, 0)$. This is indeed a consequence of Proposition 3.2 and we only have to check that $G_0(x) \cap -N_M(x) = \emptyset$ for every $x \in \partial M$. Indeed, suppose it is not true then, for some $x \in \partial M$, there exists $y \in G_0(x) \cap -N_M(x) = G(x) \cap N_M(x) \cap -N_M(x)$. Then y belongs to $N_M(x) \cap -N_M(x) = \{0\}$ since M is epi-Lipschitzian. Consequently $y = 0 \in G(x)$, a contradiction with the assumption that $0 \notin G(x)$. $\square$

3.3. A Dual Definition of the Euler Characteristic

We now state a proposition which gives an equivalent definition of the Euler characteristic, using the dual concept of tangent cone.

PROPOSITION 3.4. *Let $M \subset \mathbb{R}^n$ be compact and epi-Lipschitzian.* *(a) There exists a continuous mapping $f : M \to \mathbb{R}^n$ such that*

$$f(x) \in \operatorname{int} T_M(x) \subset T_M(x) \setminus \{0\} \text{ for every } \ x \in \partial M.$$

(b) If F is an upper semicontinuous set-valued mapping, from M to $\mathbb{R}^n$, with nonempty, convex, compact values such that

$$F(x) \subset T_M(x) \setminus \{0\} \text{ for every } \ x \in \partial M,$$

then $\chi(M) = \deg(-F, 0)$.

In the Section 4.2, we shall weaken the above tangential condition (in (b)) by only assuming that $0 \notin F(x)$ and $F(x) \cap T_M(x) \neq \emptyset$ for every $x \in \partial M$. In this section we mainly consider the case of single-valued mappings F (for which the two tangential conditions coincide).

Proof of Part (a). Since M is epi-Lipschitzian, for every $x \in M$, Clarke's normal cone $N_M(x)$ is pointed, or equivalently, $\operatorname{int} T_M(x)$ is nonempty. By Proposition 3.1.a, the set-valued mapping $x \to \operatorname{int} T_M(x)$ is lower semicontinuous, with nonempty, convex values. Consequently, by Michael's selection theorem, there exists a continuous mapping $f : M \to \mathbb{R}^n$ such that, for every $x \in M$, $f(x) \in \operatorname{int} T_M(x)$. But, for every $x \in \partial M$, $0 \notin \operatorname{int} T_M(x)$ since $T_M(x) \neq \mathbb{R}^n$ (by Proposition 2.1.b), hence $f(x) \in \operatorname{int} T_M(x) \subset T_M(x) \setminus \{0\}$. □

Part (b). This is a consequence of Proposition 3.2. We define the correspondence G, from ∂M to $\mathbb{R}^n$, by $G(x) = -F(x)$ and we check that, for every $x \in \partial M$, $G(x) \cap -N_M(x) = \emptyset$. Indeed, from the tangential condition, we get $G(x) \cap -N_M(x) \subset -[T_M(x) \setminus \{0\}] \cap -N_M(x)$, which is empty since $T_M(x) \cap N_M(x) = \{0\}$ (by Proposition 2.1). Thus, by Proposition 3.2, we get $\chi(M) = \deg(G, 0) = \deg(-F, 0)$. □

We are now able to propose an alternative definition of the Gauss set-valued mapping of a compact epi-Lipschitzian set $M \subset \mathbb{R}^n$. For every mapping $f : M \to \mathbb{R}^n$, we define

$$G_f(x) := \{p \in N_M(x) \mid p \cdot f(x) = -1\} \text{ for every } \ x \in \partial M.$$

PROPOSITION 3.5. *Let $M \subset \mathbb{R}^n$ be compact and epi-Lipschitzian and let $f : M \to \mathbb{R}^n$ be a continuous mapping satisfying*

$$f(x) \in \operatorname{int} T_M(x) \text{ for every } \ x \in \partial M.$$

Then the set-valued mapping G_f, from ∂M to $\mathbb{R}^n$, is upper semicontinuous with nonempty, convex, compact values and, for every $x \in \partial M$, $0 \notin G_f(x)$. Furthermore, $\chi(M) = \deg(G_f, 0)$.

Proof. The convexity of $G_f(x)$ is clear from the definition and the proof that $G_f(x)$ is nonempty and compact is standard from the fact that $\operatorname{int} T_M(x)$ is nonempty. The upper semicontinuity of the set-valued mapping G_f is a consequence of Proposition 3.1 and of the continuity of f. Finally, from Proposition 3.2, we get $\chi(M) = \deg(G_f, 0)$. □

We now state an important consequence of Proposition 3.4.

PROPOSITION 3.6. *Let M_1 and M_2 be compact, epi-Lipschitzian subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. Then the set $M_1 \times M_2$ is also epi-Lipschitzian and $\chi(M_1 \times M_2) = \chi(M_1) \times \chi(M_2)$.*

Proof. From Proposition 3.4.a, there are continuous mappings $f_i : M_i \to \mathbb{R}^{n_i}$ $(i = 1, 2)$ such that $f_i(x) \in \operatorname{int} T_{M_i}(x) \subset T_{M_i}(x) \setminus \{0\}$ for every $x \in \partial M_i$. We let $n = n_1 + n_2$, and we define the mapping $f : M_1 \times M_2 \to \mathbb{R}^n$ by

$$f(x_1, x_2) = (f_1(x_1), f_2(x_2)).$$

For every $(x_1, x_2) \in \partial(M_1 \times M_2) = [\partial M_1 \times M_2] \cup [M_1 \times \partial M_2]$, one clearly has $f(x_1, x_2) \neq 0$. Furthermore, from above and Clarke [11], one has

$$f(x_1, x_2) \in T_{M_1}(x_1) \times T_{M_2}(x_2) = T_{M_1 \times M_2}(x_1, x_2).$$

The above equality also shows that $\operatorname{int} T_{M_1 \times M_2}(x_1, x_2) \neq \emptyset$, hence the set $M_1 \times M_2$ is epi-Lipschitzian. From Proposition 3.4.b, we deduce that $\chi(M_1 \times M_2) = \deg(-f, 0)$, $\chi(M_i) = \deg(-f_i, 0)$, $(i = 1, 2)$ and we know that $\deg(-f, 0) = \deg(-f_1, 0).\deg(-f_2, 0)$ (see, for example [8]). Consequently, $\chi(M_1 \times M_2) = \chi(M_1).\chi(M_2)$. □

We end this section with a last consequence of Proposition 3.4.

PROPOSITION 3.7. *Let M be a convex compact subset of $\mathbb{R}^n$ with a nonempty interior. Then $\chi(M) = 1$.*

Proof. Let $\bar{x} \in \operatorname{int} M$ and let $f : M \to \mathbb{R}^n$ be defined by $f(x) = \bar{x} - x$. The mapping f is clearly continuous and, for every $x \in \partial M$, one has $f(x) \neq 0$ and $f(x) \in T_M(x) = \operatorname{cl}\{\lambda(x' - x) \mid \lambda > 0, x' \in M\}$, by Proposition 2.1.d. Consequently, from Proposition 3.4.b, we get $\chi(M) = \deg(-f, 0)$, and $\deg(-f, 0) = 1$, by Property D.3 of the degree. □

4. Fixed-point Theorems and Existence of Equilibria

4.1. Existence of generalized equilibria

We first give an existence result of generalized equilibria when the correspondence F does not satisfy a tangential condition.

PROPOSITION 4.1. *Let M be a nonempty, compact epi-Lipschitzian subset of $\mathbb{R}^n$ such that $\chi(M) \neq 0$, and let F be an u.s.c. set-valued mapping, from M to $\mathbb{R}^n$, with nonempty, compact, convex values.*

Then there exists $x^* \in M$ such that

$$0 \in F(x^*) - N_M(x^*).$$

Such an element x^* is called a generalized equilibrium of F and the above problem is sometimes called a (nonconvex) variational inequality.

Proof. Assume first that, for some $x^* \in \partial M$, $0 \in F(x^*) - N_M(x^*)$, then the result follows. Let us assume now that, for every $x \in \partial M$, $0 \notin F(x) - N_M(x)$ or, equivalently, that $-F(x) \cap -N_M(x) = \emptyset$. Then, from Proposition 3.2, we deduce that $\chi(M) = \deg(-F, 0)$, which is nonzero since $\chi(M) \neq 0$, by assumption. Consequently, from Property D.2 of the degree, there exists $x^* \in M$ such that $0 \in -F(x^*)$. Hence, $0 \in F(x^*) \subset F(x^*) - N_M(x^*)$. □

We now give a direct consequence of the previous result.

COROLLARY 4.1. *Let M be a nonempty, compact, epi-Lipschitzian subset of $\mathbb{R}^n$ such that $\chi(M) \neq 0$, and let F be an u.s.c. set-valued mapping, from M to $\mathbb{R}^n$, with nonempty, compact, convex values such that:*

(STC) $$F(x) \subset T_M(x) \text{ for every } x \in \partial M.$$

Then there exists $x^* \in M$ such that $0 \in F(x^*)$.

Proof. From Proposition 4.1, there exists $x^* \in M$ such that $0 \in F(x^*) - N_M(x^*)$, or, equivalently, such that $F(x^*) \cap N_M(x^*) \neq \emptyset$. From the Tangential Condition (STC), we deduce that

$$\emptyset \neq F(x^*) \cap N_M(x^*) \subset T_M(x^*) \cap N_M(x^*) = \{0\},$$

by Proposition 2.1.a. □

We end this section with Kakutani's fixed-point theorem.

COROLLARY 4.2 (Kakutani). *Let M be a nonempty, convex, compact, with a nonempty interior. Let Φ be an u.s.c. set-valued mapping, from M to $\mathbb{R}^n$, with nonempty, compact, convex values such that, for every $x \in \partial M$, $\Phi(x) \subset M$.*

Then, there exists $x^* \in M$ such that $x^* \in \Phi(x^*)$.

Proof. It is a consequence of Corollary 4.1, by defining the set-valued mapping F, from M to $\mathbb{R}^n$, defined by $F(x) = \Phi(x) - \{x\}$ □

4.2. Weakening the tangential condition STC

The aim of this section is to weaken the Tangential Condition in Corollary 4.1. We first prove a result, which generalizes Proposition 3.4.b.

THEOREM 4.1. *Let M be a nonempty, compact, epi-Lipschitzian subset of $\mathbb{R}^n$, and let F be an u.s.c. set-valued mapping, from M to $\mathbb{R}^n$, with nonempty, compact, convex, values, satisfying*

$$(TC) \qquad 0 \notin F(x) \text{ and } \; F(x) \cap T_M(x) \neq \emptyset \text{ for every } \; x \in \partial M.$$

Then, $\chi(M) = \deg(-F, 0)$.

Proof. We first claim that there exists a continuous mapping $f : M \to \mathbb{R}^n$ such that

$$\forall x \in \partial M, \quad 0 < \inf\{f(x) \cdot y \mid y \in F(x)\} \text{ and } f(x) \notin N_M(x).$$

Indeed, from the Tangential Condition (TC), for all $x \in \partial M$, $0 \notin F(x)$. Hence, by a separation theorem, there exists $p \in \mathbb{R}^n$ such that $0 < \inf\{p \cdot y \mid y \in F(x)\}$. For every $p \in \mathbb{R}^n$, we let

$$V(p) = \{x \in M \mid F(x) \subset \{y \in \mathbb{R}^n | 0 < p \cdot y\}\}.$$

From above, one deduces that $\partial M \subset \cup_{p \in \mathbb{R}^n} V(p)$, and, from the upper semicontinuity of F, each set $V(p)$ is open in M. Since the set ∂M is compact, there exists a finite subset $\{p_1, \ldots, p_k\}$ in $\mathbb{R}^n$ such that $\partial M \subset \cup_{i=1}^k V(p_i)$. Let $\lambda_1, \ldots, \lambda_k$ be a continuous partition of unity subordinate to the open covering $V(p_i)$, i.e., for all i, $\lambda_i : \partial M \to [0, 1]$ is continuous, $\sum_{i=1}^k \lambda_i(x) = 1$ for all x, and $\mathrm{cl}\,\{x \in \partial M \mid \lambda_i(x) > 0\}$ is a subset of $V(p_i)$. We define the mapping $f : \partial M \to \mathbb{R}^n$, by $f(x) = \sum_{i=1}^k \lambda_i(x) p_i$. Then, f is clearly continuous and we claim that

$$(\star) \qquad 0 < \inf\{f(x) \cdot y \mid y \in F(x)\} \text{ for every } \; x \in \partial M.$$

Indeed, let $x \in \partial M$ and let $\bar{y} \in F(x)$. If $\lambda_i(x) > 0$, then $x \in V(p_i)$; hence, $0 < \inf\{p_i \cdot y \mid y \in F(x)\} \leq p_i \cdot \bar{y}$. Multiplying each inequality by $\lambda_i(x)$ and summing up, one gets

$$0 < \sum_{i=1}^{k} \lambda_i(x) \inf\{p_i \cdot y \mid y \in F(x)\} \leq f(x) \cdot \bar{y},$$

which, together with the compactness of $F(x)$, implies $(\star)$.

We now show that, for every $x \in \partial M$, $f(x) \notin N_M(x)$. Indeed, suppose that, for some $x \in \partial M$, $f(x) \in N_M(x)$, from the Tangential Condition (TC), there is some $\bar{y} \in F(x) \cap T_M(x)$ and, from $(\star)$, we deduce that

$$0 < \inf\{f(x) \cdot y \mid y \in F(x)\} \leq f(x) \cdot \bar{y} \leq 0,$$

a contradiction. This ends the proof of the claim.

We now end the proof of the theorem and we will show successively that $\chi(M) = \deg(-f, 0)$ and $\deg(-f, 0) = \deg(-F, 0)$. The first equality is a consequence of Proposition 3.2 and the fact that, for every $x \in \partial M$, $-f(x) \notin -N_M(x)$ by the above claim. We now prove that $\deg(-f, 0) = \deg(-F, 0)$, and in view of Property D.5 of the degree, it suffices to check that $0 \notin \text{co}\ [\{-f(x)\} \cup -F(x)]$ for every $x \in \partial M$. Indeed, if it is not true, then $0 = (1-t)f(x) + t\bar{y}$ for some $x \in \partial M$, $t \in [0, 1]$ and $\bar{y} \in F(x)$. One notices that $t \neq 1$ since $0 \notin F(x)$. Consequently, from $(\star)$, we get

$$0 < \inf\{f(x) \cdot y \mid y \in F(x)\} \leq f(x) \cdot \bar{y} = -\tfrac{t}{1-t}\bar{y} \cdot \bar{y} \leq 0,$$

a contradiction. □

We now give a generalization of Corollary 4.1.

COROLLARY 4.3. *Let M be a nonempty, compact, epi-Lipschitzian subset of $\mathbb{R}^n$ such that $\chi(M) \neq 0$. Let F be an u.s.c. set-valued mapping, from M to $\mathbb{R}^n$, with nonempty, compact, convex values such that:*

(TC) $\qquad F(x) \cap T_M(x) \neq \emptyset$ for every $x \in \partial M$.

Then, there exists $x^* \in M$ such that $0 \in F(x^*)$.

Proof. Assume first that, for some $x^* \in \partial M$, $0 \in F(x^*)$, then the result follows. Let us assume now that, for every $x \in \partial M$, $0 \notin F(x)$. From Theorem 4.1, we deduce that $\chi(M) = \deg(-F, 0)$, which is nonzero since $\chi(M) \neq 0$, by assumption. Consequently, from Property D.2 of the degree, there exists $x^* \in M$ such that $0 \in F(x^*)$. $\square$

5. Euler Characteristic and Lipschitz Representations of M

We recall that every nonempty compact epi-Lipschitzian set $M \subset \mathbb{R}^n$ can be written as follows $M = \{x \in \mathbb{R}^n \mid \varphi(x) \leq 0\}$, for some locally Lipschitzian function $\varphi : \mathbb{R}^n \to \mathbb{R}$, satisfying a Nondegenerate Condition. The main aim of this section is to show the relation between the Euler characteristic $\chi(M)$ and such functions φ.

5.1. Nondegenerate Lipschitz Representation of the set M

We first recall some definitions. A function φ is said to be locally Lipschitz if, for every $x \in \mathbb{R}^n$, there exist a neighborhood N of x and a positive real number k such that $|\varphi(x_1) - \varphi(x_2)| \leq k\|x_1 - x_2\|$ for every x_1 and x_2 in N. Following Clarke (1975) we define the generalized gradient $\partial\varphi(x)$ of φ, at a point $x \in \mathbb{R}^n$, to be the set

$$\partial\varphi(x) = \operatorname{cl}\operatorname{co}\{\lim_{\nu} \nabla\varphi(x^\nu) \mid (x^\nu) \subset D(\nabla\varphi), (x^\nu) \to x\},$$

where $D(\nabla\varphi)$ denotes the set on which φ is differentiable and $\nabla\varphi(x^\nu)$ denotes the gradient of φ at x^ν. We recall that the set-valued mapping $x \to \partial\varphi(x)$ is u.s.c., with nonempty, convex, compact values and that, if φ is continuously differentiable on a neighborhood of a point x_0, then $\partial\varphi(x_0) = \{\nabla\varphi(x_0)\}$. For these definitions and properties, we refer to Clarke (1983).

We now consider the class of nonempty, compact subsets M of $\mathbb{R}^n$ which can be written as follows

$$M = \{x \in \mathbb{R}^n \mid \varphi(x) \leq 0\},$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function, which satisfies the following Nondegeneracy Condition

$$(ND) \qquad 0 \notin \partial\varphi(x) \ \text{ if } \ \varphi(x) = 0,$$

and we shall say that φ is a nondegenerate Lipschitz representation of the set M, or simply a representation of M. We denote by $\mathcal{R}(\mathcal{M})$ the set of all the representations φ of M.

We recall that M is epi-Lipschitzian if and only if it admits a representation.

PROPOSITION 5.1. *Let $M \subset \mathbb{R}^n$ be nonempty and compact. Then the two following assertions are equivalent:*

(i) M is epi-Lipschitzian;

(ii) $\mathcal{R}(\mathcal{M}) \neq \emptyset$;

Moreover, for every $\varphi \in \mathcal{R}(\mathcal{M})$, $\partial M = \{x \in \mathbb{R}^n \mid \varphi(x) = 0\}$ and

$$N_M(x) \subset \bigcup_{\lambda \geq 0} \lambda\, \partial\varphi(x) \quad \textit{if} \;\; x \in \partial M.$$

The assertion $[(ii) \Rightarrow (i)]$ is proved in Clarke (1983) together with the above inclusion of the normal cone. The assertion $[(i) \Rightarrow (ii)]$ is proved in Cornet-Czarnecki [19], by considering the following function $\Delta_M : \mathbb{R}^n \to \mathbb{R}$ defined by $\Delta_M(x) = d_M(x) - d_{\mathbb{R}^n \setminus M}(x)$, where d_M denotes the distance function to M.

The next proposition relates the Euler characteristic of M with degree of the set-valued mapping $x \to \partial\varphi(x)$.

PROPOSITION 5.2. *Let $M \subset \mathbb{R}^n$ be nonempty, compact and epi-Lipschitzian and let $\varphi \in \mathcal{R}(\mathcal{M})$, then $\chi(M) = \deg(\partial\varphi(\cdot), 0)$.*

Proof. Let $\varphi \in \mathcal{R}(\mathcal{M})$, from Proposition 5.1, one has

$$N_M(x) \subset \bigcup_{\lambda \geq 0} \lambda\, \partial\varphi(x) \quad \text{for every} \;\; x \in \partial M.$$

Recalling that, for every $x \in \partial M$, $N_M(x) \neq \{0\}$ [Proposition 2.1], one deduces that

$$0 \notin \partial\varphi(x) \text{ and } \;\partial\varphi(x) \cap N_M(x) \neq \emptyset \quad \text{for every} \;\; x \in \partial M.$$

Consequently, from Proposition 3.3, recalling that the set-valued mapping $x \to \partial\varphi(x)$ is u.s.c., with nonempty, convex, compact values, one gets $\chi(M) = \deg(\partial\varphi(\cdot), 0)$. □

REMARK 5.1. We point out the following important case, where we can get an explicit representation of M, thus compute the degree of its generalized gradient. Consider a set M defined by finitely many smooth inequalities, that is, M can be written as follows

$$M = \{x \in \mathbb{R}^n \mid \varphi_i(x) \leq 0 \quad \forall i \in I\},$$

where I is a finite set and $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ $(i \in I)$ are continuously differentiable functions which satisfy the following Constraint Qualification Assumption

Assumption CQ: For every $x \in M$, the family $\{\nabla\varphi_i(x) \mid i \in I(x)\}$ is independent, where $I(x) = \{i \in I \mid \varphi_i(x) = 0\}$ denotes the set of binding constraints.

Under the above assumptions, one shows [see, for example [11]] that the function $\varphi : \mathbb{R}^n \to \mathbb{R}$ defined by $\varphi(x) = \sup_{i\in I}\varphi_i(x)$ is a representation of the set M and

$$\partial\varphi(x) = \mathrm{co}\,\{\nabla\varphi_i(x) \mid i \in I(x)\}.$$

REMARK 5.2. If the set M is nonempty, closed, and $M \neq \mathbb{R}^n$, the function Δ_M defined above is clearly Lipschitzian on $\mathbb{R}^n$ and satisfies:

$$M = \{x \in \mathbb{R}^n | \Delta_M(x) \leq 0\} \text{ and } \partial M = \{x \in \mathbb{R}^n | \Delta_M(x) = 0\}.$$

If we additionally assume that M is epi-Lipschitzian, then

$$0 \notin \partial\Delta_M(x) \subset \mathrm{co}\,[N_M(x) \cap S] := G_M(x) \text{ for all } x \in \partial M,$$

thus $\Delta_M \in \mathcal{R}(\mathcal{M})$ (see [19]). It is worth pointing out that, what was done above with the function Δ_M, cannot be done with the the distance function d_M. Indeed, an arbitrary closed set $M \subset \mathbb{R}^n$ can always be written as $M = \{x \in \mathbb{R}^n \mid \ d_M(x) \leq 0\}$, but the above Nondegeneracy Condition never holds with $\varphi = d_M$ since, for all $x \in M$, $0 \in \partial d_M(x)$.

5.2. A class of sets with a nonzero Euler characteristic

The main result of this section gives a sufficient condition on the representation $\varphi \in \mathcal{R}(\mathcal{M})$ which guarantees that $\chi(M) = 1$. This result comes from the study of general equilibrium theory with nonconvex production sets [3, 4, 6, 15].

THEOREM 5.1. *Let M be nonempty, compact subset of $\mathbb{R}^n$, and let $\varphi \in \mathcal{R}(\mathcal{M})$ such that $0 \notin \partial\varphi(x)$ if $\varphi(x) > 0$. Then $\chi(M) = 1$.*

The proof of Theorem 5.1 will use the following lemma.

LEMMA 5.1 (Bonnisseau–Cornet [5]). *Under the assumptions of Theorem 5.1, there exist $\varepsilon > 0$ and a continuous mapping r from $\mathbb{R}^n$ to $M_\varepsilon := \{x \in \mathbb{R}^n \mid \varphi(x) \leq -\varepsilon\}$ satisfying the two following conditions*

(i) $r(x) = x$ for every $x \in M_\varepsilon$;

(ii) $0 > \sup\{p \cdot (r(x) - x) \mid p \in \partial\varphi(x)\}$ if $\varphi(x) = 0$.

Proof of Theorem 5.1. We let $\varepsilon > 0$ and r be defined as in the above lemma. Let B be a closed ball containing a neighborhood of the set M, we define the mapping $f : B \to \mathbb{R}^n$ by $f(x) = x - r(x)$.

From the Excision Property D.4 of the degree, since the set $K = B \setminus \operatorname{int} M$ is compact, and $f(x) \neq 0$ for all $x \in K$, one deduces that $\deg(f, 0) = \deg(f_{|M}, 0)$, where $f_{|M}$ denotes the restriction of the mapping f to M.

The proof of the theorem consists to show the two following assertions, first, $1 = \chi(B) = \deg(f, 0)$ and, second, $\chi(M) = \deg(f_{|M}, 0)$.

The first assertion $\chi(B) = 1$ comes from the convexity of B by Proposition 3.7. Furthermore, $\chi(B) = \deg(f, 0)$ from Proposition 3.4.b, since the continuous mapping $f : B \to \mathbb{R}^n$ clearly satisfies

$$\forall x \in \partial B, \; -f(x) := r(x) - x \in T_B(x) \setminus \{0\}.$$

The proof that $\chi(M) = \deg(f_{|M}, 0)$ is also a consequence of Proposition 3.4.b and we only need to show that the continuous mapping $f_{|M} : M \to \mathbb{R}^n$ satisfies

$$\forall x \in \partial M, \; -f_{|M}(x) := r(x) - x \in T_M(x) \setminus \{0\}.$$

Indeed, we first notice that for every $x \in \partial M$, $r(x) - x \neq 0$ since $r(x) \in M_\varepsilon$ which does not meet ∂M. Since $\varphi \in \mathcal{R}(\mathcal{M})$ is a representation of M, from Proposition 5.1, we deduce that

$$N_M(x) \subset \bigcup_{\lambda \geq 0} \lambda \; \partial\varphi(x).$$

Consequently, from Condition (ii) of Lemma 5.1 and the bipolar theorem we deduce that

$$r(x) - x \in \partial\varphi(x)^0 \subset N_M(x)^0 = T_M(x).$$

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