Fatou's Lemma for Gelfand Integrable Mappings

B. CORNET
CERMSEM, Maison des Sciences Economiques, Université Paris I, 106–112 Boulevard de l'Hopital, 75645 Paris Cedex 13, France.
E-mail: cornet@univ-paris1.fr

J.-P. MEDECIN
CERMSEM, Maison des Sciences Economiques, Université Paris I, 106–112 Boulevard de l'Hopital, 75645 Paris Cedex 13, France.
E-mail: medecin@univ-paris1.fr

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Abstract. We provide a version of Fatou's lemma for mappings taking their values in E^*, the topological dual of a separable Banach space. The mappings are assumed to be Gelfand integrable, a difference with previous papers, which, in infinite dimensional spaces, are mainly considering Bochner integrable mappings. This result is motivated by a general equilibrium model with locations studied by Cornet–Medecin [13] and directly applies to it, since the space E^* considered in [13] is the space of (Radon) vector measures defined on a compact metric space.

Keywords: Fatou's lemma, Banach space, dual space, measure space, weak-star integral, Gelfand integral.

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1. Introduction

We provide a version of Fatou's lemma for mappings taking their values in E^*, the topological dual of a separable Banach space. The mappings are assumed to be Gelfand integrable, also called weak-star integrable, a main difference with previous papers, which, in infinite dimensional spaces, were considering other types of integrable mappings. Versions of Fatou's lemma with several (finite) dimensions were first proved by Schmeidler [26], Hildenbrand and Mertens [22], Arstein [3], and Balder [5, 6]. These results were extended to the case of infinitely many dimensions by Khan and Majumdar [23], Balder [6], Yannelis [28, 29], Papageorgiou [24], Balder and Hess [7], and Castaing and Saadoune [11]. These papers are either considering Bochner integrable mappings or Pettis integrable mappings, whereas here we shall consider Gelfand integrable mappings.

The choice of Gelfand integrable mappings and the statement of Fatou's lemma given here are directly motivated by an economic model of spatial economies developed in a companion paper (see Cornet and Medecin [13]). This model considers a measure space of agents, each one one choosing a consumption in the commodity space $\mathcal{M}(K)$, the space of (Radon) measures defined on a compact metric space K, a framework, which is clearly a particular case of the one considered here. Our paper [13] follows the approach initiated in economics in the 70's (see, for example, Hildenbrand [21]) for the existence of equilibria in an economy with a measure space of agents. In particular, the existence proof relies on a version of Fatou's lemma, which needs to be formulated in $\mathcal{M}(K)$ (or more generally, in the dual of a separable Banach space) to be able to consider infinitely many locations, that is, allowing the set K to be infinite, more precisely to be compact and metrizable.

2. Statement of Fatou's lemma

2.1. Preliminaries on Gelfand integration

Throughout this paper, we let $(\Omega, \mathcal{A}, \mu)$ be a complete finite positive measure space, and we let $L^1(\Omega, \mathcal{A}, \mu; \mathbb{R})$, or simply $L^1(\mu)$, be the set of equivalence classes (for the equivalence relation $f(a) = g(a)$ for a.e. $a \in \Omega$) of integrable functions $f : \Omega \to \mathbb{R}$. We shall also consider a separable Banach space, with topological dual space E^*, that is, E^* is the set of all continuous linear functions $x^* : E \to \mathbb{R}$.[1]

We need now to consider "integrable" mappings $f : \Omega \to E^*$, and thus recall some definitions. A mapping $f : \Omega \to E^*$ is said to be Gelfand measurable, also called weak-star measurable, or E-scalarly measurable (resp. Gelfand integrable, also called weak-star integrable, or E-scalarly integrable) if, for every $x \in E$, the real-valued function $a \mapsto \langle x, f(a)\rangle$, from Ω to $\mathbb{R}$, is measurable (resp. integrable). If a mapping $f\colon \Omega \to E^*$ is Gelfand integrable, then for every $A \in \mathcal{A}$, we define the real-valued function $\int_A f(a)d\mu(a) : E \to \mathbb{R}$, by

$$\Big[\int_A f(a)d\mu(a)\Big](x) = \int_A \langle x, f(a)\rangle d\mu(a),\ \forall x \in E.$$

[1] For every $x \in E, f \in E^*$ we denote by $\langle x, f\rangle := f(x)$ the dual pair, by $\|x\|$, the norm of E, by $\|f\| := \sup\{|\langle x, f\rangle| \mid x \in E, \|x\| \leq 1\}$ the dual norm of E^* and we let $B = \{x \in E \mid \|x\| \leq 1\}$, $B_* = \{f \in E^* \mid \|f\| \leq 1\}$ be the closed unit balls of E and E^*, respectively. In most of the paper, the space E^* will be endowed with the weak-star topology, denoted $\sigma(E^*, E)$ and we shall denote by cl C, lim, etc., the closure of a subset C of E^*, the limit, for the weak-star topology. In some cases, we shall also consider on E^*, the topology defined by the dual norm (also called the strong topology) and we shall denote by $s -$ cl C, $s -$ lim, etc., the closure of a subset C of E^*, the limit, for the strong topology.

Clearly, $\int_A f(a)d\mu(a)$ is linear (from the linear property of the integral on $\mathbb{R}$) and it is also continuous[2] (see Diestel and Uhl [17, Theorem p. 52 and remarks]). For every $A \in \mathcal{A}$, the element $\int_A f(a)d\mu(a)$, which thus belongs to E^*, is called the Gelfand integral of f on A. We note $L^1_G(\Omega, \mathcal{A}, \mu; E^*)$ (or $L^1_G(E^*)$) the set of equivalence classes of Gelfand integrable mappings $f : \Omega \to E^*$.

2.2. Fatou's lemma with Gelfand integral

Let (x_n) be a sequence in E^*. We denote by $\mathrm{Ls}_n\{x_n\}$ the set of weak-star limit points of converging subsequences of (x_n), i.e.,

$$\mathrm{Ls}_n\{x_n\} := \{x \in E^* |\ \exists \varphi \in \mathcal{I},\ x = \lim_k x_{\varphi(k)}\},$$

where $\mathcal{I} = \{\varphi : \mathbb{N} \to \mathbb{N} |\ \varphi$ increasing, i.e., $\varphi(k+1) > \varphi(k)$ for all $k\}$.

We now state the main result of the paper.

THEOREM 1. *Let $(\Omega, \mathcal{A}, \mu)$ be a complete finite positive measure space, let E be a separable Banach space, and let (f_n) be a sequence of Gelfand integrable mappings from Ω to E^* such that*

(i) $\lim_n \int_\Omega f_n(a)d\mu(a)$ *exists;*

(ii) the sequence (f_n) is integrably bounded, in the sense that there exists an integrable function $g : \Omega \to \mathbb{R}$ such that, for a.e. $a \in \Omega$, $\sup_n \|f_n(a)\| \leq g(a)$.

(a) [Convex Fatou Lemma]. *There exists a Gelfand integrable mapping $f : \Omega \to E^*$ such that:*

$$\int_\Omega f(a)d\mu(a) = \lim_n \int_\Omega f_n(a)d\mu(a); \tag{1}$$

$$\textit{for a.e. } a \in \Omega,\ f(a) \in \mathrm{cl\ co\ Ls}_n\{f_n(a)\}. \tag{2}$$

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[2] For the sake of completeness, we provide hereafter the (classical) proof of the continuity, for every $A \in \mathcal{A}$, of the linear mapping $x \mapsto \int_A \langle x, f(a)\rangle d\mu(a)$. We define the linear mapping $T : E \to L^1(\mu)$ by $T(x) : a \to \langle x, f(a)\chi_A(a)\rangle$ for every $x \in E$ (where $\chi_A : \Omega \to \mathbb{R}$ is defined by $\chi_A(a) = 1$ if $a \in A$ and $\chi_A(a) = 0$ elsewhere). We claim that the graph of T, i.e., $GT := \{(x, g) \in E \times L^1(\mu) |\ g = T(x)\}$, is closed in $E \times L^1(\mu)$ for the product of the norm-topologies. Indeed, let $(x_n, T(x_n))$ be a sequence in $E \times L^1(\mu)$ converging to some element $(x, g) \in E \times L^1(\mu)$. Then, there exists a subsequence $(T(x_{n_k}))$ such that, for a.e. $a \in \Omega$, $(T(x_{n_k})(a))$ converges to $g(a)$ (see, for example, [18, Theorem 6, p. 122 and Corollary 13 p. 150]). Consequently, for a.e. $a \in \Omega$, $g(a) = \lim_k \langle x_{n_k}, \chi_A(a)f(a)\rangle = \langle x, \chi_A(a)f(a)\rangle = T(x)(a)$, i.e., $T(x) = g$. This ends the proof that T has a closed graph. Consequently, by the Closed graph theorem (see, for example, [18, Theorem 4, p. 57], the linear mapping T is continuous. Since the mapping $h \mapsto \int_\Omega h d\mu(a)$ from $L^1(\mu)$ to $\mathbb{R}$ is a continuous linear mapping with norm at most 1, then $|\int_A \langle x, f(a)\rangle d\mu(a)| \leq \|T(x)\| \leq \|T\|\ \|x\|$. Thus, the mapping $x \mapsto \int_A \langle x, f(a)\rangle d\mu(a)$ is continuous.

(b) [Approximate Fatou Lemma]. *For every neighborhood W of 0 in E^* (for the weak-star topology), there exists a Gelfand integrable mapping $f_W\colon\Omega\to E^*$ such that*

$$\int_\Omega f_W(a)d\mu(a)-\lim_n \int_\Omega f_n(a)d\mu(a)\in W\ ; \tag{3}$$

$$\textit{for a.e. } a\in\Omega,\ f_W(a)\in \mathrm{Ls}_n\{f_n(a)\}. \tag{4}$$

[Exact Fatou Lemma]. *If E is additionally assumed to be finite dimensional, there exists a Gelfand integrable mapping $f\colon\Omega\to E^*$ such that*

$$\int_\Omega f(a)d\mu(a)=\lim_n \int_\Omega f_n(a)d\mu(a)\ ; \tag{5}$$

$$\textit{for a.e. } a\in\Omega,\ f(a)\in \mathrm{Ls}_n\{f_n(a)\}. \tag{6}$$

The proof of Theorem 1 is given in Section 3. We end this section with several remarks.

REMARK 1. The above version of Fatou's lemma is stated under the assumption that the sequence (f_n) is integrably bounded, and this is sufficient for what is needed in the economic model considered by Cornet–Medecin [13]. Finite dimensional version of Fatou's lemma were proven by Schmeidler [26], Hildenbrand and Mertens [22], Arstein [3], and Balder [6]. Some of these papers [3, 6] are considering the weaker assumption that the sequence (f_n) is uniformly integrable in the sense that, for (x,n) in $B\times\mathbb{N}$, the family of real-valued functions $\langle x, f_n(\cdot)\rangle : a\mapsto\langle x, f_n(a)\rangle$ is uniformly integrable, that is,

$$\lim_{q\to\infty}\Big[\sup_{\substack{n\in\mathbb{N}\\ x\in B}}\int_{|\langle x,f_n(.)\rangle|>q}|\langle x,f_n(a)\rangle|d\mu(a)\Big]=0.$$

REMARK 2. When E is infinite dimensional, the exact assertion (5) may not be true. We refer to Rustichini [25] for a counterexample.

2.3. Fatou's lemma with Bochner integrals

To allow the comparison between the choice of Gelfand and Bochner integrals, we now state a version of Fatou's lemma for Bochner integrable mappings. We assume below (and only below) that the space E is a *reflexive* separable Banach space. This clearly excludes the space $\mathcal{M}(K)$ of (Radon) vector measure on a compact metric space K, considered in Cornet–Medecin [13].

We first recall the definition Bochner's integrable and we refer, for example to [30] for the properties of the integral.[3] A mapping $f : \Omega \to E^*$ is said to be finitely-valued (also called simple) if it is constant on each of a finite number of disjoint measurable sets. A mapping $f : \Omega \to E^*$ is said to be Bochner integrable, if (i) it is strongly measurable, in the sense that there is a sequence of finitely-valued mappings $f_n : \Omega \to E^*$ converging to f almost everywhere on Ω, for the norm topology of E^*, and (ii) the sequence (f_n) satisfies $\lim_n \int_\Omega \|f(a) - f_n(a)\| d\mu(a) = 0$. Then one shows that, for every $A \in \mathcal{A}$, $\lim_n \int_A f_n(a) d\mu(a)$ exists and is independent of the choice of the sequence (f_n). We then let the Bochner's integral of f on A to be $\int_A f(a) d\mu(a) := \lim_n \int_A f_n(a) d\mu(a)$.

THEOREM 2. [Approximate Bochner–Fatou Lemma]. *Let $(\Omega, \mathcal{A}, \mu)$ be a complete finite positive measure space, let E be a reflexive separable Banach space, and let (f_n) be a sequence of Bochner integrable mappings from Ω to E^* such that*

(i) $\lim_n \int_\Omega f_n(a) d\mu(a)$ exists;

(ii) the sequence (f_n) is integrably bounded, in the sense that, there exists an integrable function $g : \Omega \to \mathbb{R}$ such that, for every n, for a.e. $a \in \Omega$, $\|f_n(a)\| \leq g(a)$.

Then, for every $\epsilon > 0$, there exists a Bochner integrable mapping $f_\epsilon : \Omega \to E^*$ such that:

$$\left\| \int_\Omega f_\epsilon(a) d\mu(a) - \lim_n \int_\Omega f_n(a) d\mu(a) \right\| \leq \epsilon; \tag{7}$$

$$\text{for a.e. } a \in \Omega, \; f_\epsilon(a) \in \mathrm{Ls}_n\{f_n(a)\}. \tag{8}$$

The proof of Theorem 2 is given in Section 3. For more general result in this direction, we refer to Khan and Majumdar [23], Balder [6], Yannelis [27, 28], Papageorgiou [24], Balder and Hess [7].

REMARK 3. Theorem 2 may no longer be true if E is not assumed to be reflexive. Indeed, let $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B}[0, 1]$, be the Borel σ-algebra, μ be the Lebesgue measure on $\mathbb{R}$, $E = C[0, 1]$ be the set of

[3] To remain within the same framework considered in the previous section, we consider Bochner integrable mappings with values in E^*. But the same results hold also for mappings with values in E, since throughout the section we assume that the Banach space E is reflexive and separable, which is equivalent to saying that E^* is reflexive and separable.

continuous real-valued functions on $[0,1]$, and $E^* = \mathcal{M}[0,1]$, the space of Radon measures on $[0,1]$. For $t \in \mathbb{R}$, the Dirac (Radon) measure $\delta_t : C[0,1] \to \mathbb{R}$ is defined by $\delta_t(f) = f(t)$ and we let $[t]$ denote the integer part of t. We consider the sequence of mappings (f_n) from $[0,1]$ to E^* defined by

$$f_n(a) = \delta_{([2^n a]+\frac{1}{2})/2^n} \text{ for every } n \text{ and every } a \in [0,1).$$

We prove hereafter the five following assertions (i)-(v), which taken altogether, show that Theorem 2 does not hold for $E^* = \mathcal{M}[0,1]$.

- (i) for all n, f_n is Bochner integrable;
- (ii) for all n, $a \in [0,1)$, $\|f_n(a)\| = 1$, hence f_n is integrably bounded;
- (iii) $\| \int_{[0,1]} f_n(a) d\mu(a)\| \leq 1$. Hence, from the Alaoglu–Bourbaki theorem, without any loss of generality, we can additionally assume that $\lim_n \int_{[0,1]} f_n(a) d\mu(a)$ exists;
- (iv) for every $a \in [0,1)$, $\lim_n \ f_n(a) = \delta_a$;
- (v) the mapping $a \mapsto \delta_a$ is Gelfand integrable but is not Bochner integrable.

Proof. (i) One remarks that for every n,

$$f_n = \sum_{i=0}^{2^n-1} \chi_{[i/2^n,(i+1)/2^n)} \delta_{(i+\frac{1}{2})/2^n}.$$

Since for every n and every $i = 0, \ldots, 2^n$, $[i/2^n, (i+1)/2^n) \in \mathcal{A}$, the mapping f_n is finitely-valued and measurable (also called a simple mapping). Hence, it is Bochner integrable.

(ii) For every n, every $a \in [0,1)$ and every $x \in C[0,1]$,

$$\langle x, f_n(a)\rangle = x(([2^n a] + \tfrac{1}{2})/2^n) \leq \|x\|_\infty.$$

Consequently, $\|f_n(a)\| \leq 1$. By considering the constant mapping x equal to 1, we deduce that $\|f_n(a)\| = 1$.

(iii) From (ii), we get $\| \int_{[0,1]} f_n(a) d\mu(a)\| \leq \int_{[0,1]} \|f_n(a)\| d\mu(a) \leq 1$.

(iv) One has $([2^n a] + \frac{1}{2})/2^n \in (a - 1/2^{n+1}, a + 1/2^{n+1}]$ for every n and every $a \in [0,1)$. For every $x \in C[0,1]$, we notice that the sequence $\langle x, f_n(a)\rangle = x(([2^n a] + \frac{1}{2})/2^n)$ converges to $x(a) = \langle x, \delta_a\rangle$. Hence $\lim_n f_n(a) = \delta_a$.

(v) Indeed, it is clearly Gelfand integrable since for every $x \in C[0,1]$, the mapping $a \mapsto \langle x, \delta_a\rangle = x(a)$ is integrable (since x is continuous).

We now show that f is not strongly measurable (hence it is not Bochner integrable) by contraposition. Suppose that it is strongly measurable, then there exists a sequence of finitely-valued, measurable mappings $(f_n)_{n\in\mathbb{N}}$ such that, for a.e. $a \in [0,1]$, $f(a) = \text{s-lim}_n f_n(a)$. By Egoroff's Theorem,[4] there exists a measurable subset $A \subset [0,1]$ such that $\mu(A) > 0$, and (f_n) converges to f uniformly on A. Hence, there exists n such that $\sup_{a\in A} \|f(a) - f_n(a)\| \leq \frac{1}{2}$. But the function f_n is finitely-valued, that is, $f_n = \sum_{i\in I_n} u_i^n \chi_{A_i^n}$ for some finite set I_n, some measurable partition $(A_i^n)_{i\in I_n}$ of $[0,1]$ and some $u_i^n \in \mathcal{M}[0,1] (i \in I_n)$. We choose $i \in I_n$ such that $\mu(A_i^n \cap A) > 0$, and we choose a, a' in $A_i^n \cap A$, $a \neq a'$. Then we have $f_n(a) = f_n(a')$, hence

$$\|f(a) - f(a')\| \leq \|f(a) - f_n(a')\| + \|f_n(a') - f(a')\| \leq \tfrac{1}{2} + \tfrac{1}{2}.$$

This contradicts the fact that $\|f(a) - f(a')\| = 2$. To show this, recall that $f(a) = \delta_a$, that $\|\delta_a - \delta_{a'}\| = \sup_{x\in C[0,1]} \|x(a) - x(a')\|$, and choose $x : [0,1] \to \mathbb{R}$ continuous such that $x(a) = +1$ and $x(a') = -1$. □

3. Proof of Fatou's lemma

The next section prepares the proof of Fatou's lemma with some results (more or less known) for which we provide a proof for the sake of completeness.

3.1. Sequentially weakly compact subsets of $L^1_G(E^*)$

Let $(\Omega, \mathcal{A}, \mu)$ be a complete finite positive measure space. We say that a sequence (f_n) of Gelfand integrable mappings from Ω to E^* converges weakly to a Gelfand integrable mapping $f : \Omega \to E^*$ if, for every $x \in E$, the sequence of (real-valued) functions $a \mapsto \langle x, f_n(a)\rangle$ converges to the function $a \mapsto \langle x, f(a)\rangle$ for the weak topology of $L^1(\mu)$, i.e., the topology $\sigma(L^1(\mu), L^\infty(\mu))$. Since the simple functions are dense in $L^\infty(\mu)$ for the norm topology, it is equivalent to the following assertion (see, for example, [18, Theorem 7, p. 1291]),

$$\forall x \in E,\ \forall A \in \mathcal{A},\ \lim_{n\to\infty} \int_A \langle x, f_n(a)\rangle d\mu(a) = \int_A \langle x, f(a)\rangle d\mu(a),$$

[4] Let E be a separable Banach space, let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, and let (f_n) be a sequence of Gelfand measurable mappings from Ω to E^* converging strongly a.e. on Ω to a Gelfand measurable mapping $f : \Omega \to E^*$. Then, for every $\varepsilon > 0$, there exists a measurable subset $A \in \mathcal{A}$ such that $\mu(\Omega \setminus A) < \varepsilon$, and the sequence (f_n) converges to f uniformly on A. The proof of the above assertion is the same as for the classical statement of Egoroff's Theorem, i.e., for $E = \mathbb{R}$, noticing that the proof (when $E = \mathbb{R}$) only uses the fact that the real-valued function $a \to \|f_n(a) - f(a)\|$ is measurable, which is clearly the case when both f_n and f are Gelfand measurable and E is separable.

together with the fact that the sequence (f_n) is bounded.

The following result, which is a key tool in the proof of Fatou's lemma, gives a sufficient condition for weak sequential compactness in $L^1_G(\Omega, \mathcal{A}, \mu; E^*)$, which can be compared with the classical result of Dunford ([17, Theorem 15, p. 77], [16]) when $E = \mathbb{R}$.

THEOREM 3. [Weak sequential compactness in $L^1_G(\Omega, \mathcal{A}, \mu; E^*)$] *Let (f_n) be a sequence of Gelfand integrable mappings from Ω to E^*, which is integrably bounded. Then, the sequence (f_n) is weakly sequential compact, that is, there exists a subsequence (f_{n_k}), which converges weakly to some Gelfand integrable mapping $f : \Omega \to E^*$.*

For the proof of the theorem, we shall use the following version of the Radon–Nikodym theorem for vector measures (a version of the Radon–Nikodym theorem for correspondences is given in Debreu and Schmeidler [18]). We say that a mapping $\phi : \mathcal{A} \to E^*$ is a vector measure if, for every $x \in E$, the function $A \mapsto \langle x, \phi(A)\rangle$, from $\mathcal{A}$ to $\mathbb{R}$, is a σ-additive measure. It is said to be bounded if $\sup_{A\in\mathcal{A}} \|\phi(A)\| < \infty$, and it is said to be absolutely continuous with respect to the measure μ if $\mu(A) = 0$ implies $\phi(A) = 0$.

THEOREM 4. [Radon–Nikodym] *Let $(\Omega, \mathcal{A}, \mu)$ be a complete finite positive measure space, let E be a separable Banach space, and let ϕ be a mapping from $\mathcal{A}$ to E^*. Then, the following two assertions are equivalent:*

(i) ϕ is a bounded vector measure which is absolutely continuous with respect to μ;

(ii) there exists a Gelfand integrable mapping $f : \Omega \to E^*$ such that

$$\forall A \in \mathcal{A},\ \phi(A) = \int_A f(a)d\mu(a).$$

For the proof of Theorem 4, see Diestel and Uhl [17, Remark 4, p. 83].[5] We now come back to the proof of Theorem 3.

[5] For the sake of completeness, we provide the proof of this result. Let ϕ be a bounded vector measure which is absolutely continuous with respect to μ. We define the linear mapping T : vect $\{\chi_A |\ A \in \mathcal{A}\} \to E^*$ by $T(\chi_A) = \phi(A)$. Since the set of simple functions is dense in $L^1(\mu)$, by an extension theorem, there exists a linear mapping $\hat{T} : L^1(\mu) \to E^*$, which coincides with T on vect $\{\chi_A |\ A \in \mathcal{A}\}$. From Diestel and Uhl [17, Remark 4, p. 83], there exists a Gelfand integrable mapping $f : \Omega \to E^*$ such that

$$\forall x \in E,\ \forall g \in L^1(\mu),\ \langle x, \hat{T}(g)\rangle = \int_\Omega \langle x, g(a)f(a)\rangle d\mu(a).$$

Proof of Theorem 3. For every n and every $A \in \mathcal{A}$, we let:

$$\phi_n(A) = \int_A f_n(a) d\mu(a).$$

From the Radon–Nikodym theorem (Theorem 4), we know that the mapping $\phi_n: \mathcal{A} \to E^*$ is a bounded vector measure. We need to prove the following claim.

CLAIM 1. *There exists a subsequence (ϕ_{n_k}) such that, for every A in $\mathcal{A}$, $(\phi_{n_k}(A))$ (weak-star) converges to some element $\phi(A) \in E^*$.*

Proof of Claim 1. Let D be a countable dense subset of E. Since, for every $x \in D$, and every n we have

$$\forall A \in \mathcal{A},\ |\langle x, \phi_n(A)\rangle| \leq \int_A |\langle x, f_n(a)\rangle| d\mu(a) \leq \|x\| \int_A g(a) d\mu(a), \quad (9)$$

the set $H_x = \{\langle x, f_n(.)\rangle |\ n \in \mathbb{N}\} \subset L^1(\mu)$ is uniformly integrable, that is, H_x is bounded and $\lim_{\mu(A)\to 0} \sup_{h\in H_x} \int_A |h(a)| d\mu(a) = 0$.

By Dunford's theorem [18, Theorem 9, p. 282], for every $x \in D$, the subset H_x is weakly compact in $L^1(\mu)$ and, from the Eberlein–Smulian theorem [18, Theorem 1, p. 430] there exists a subsequence of $(\langle x, f_n(.)\rangle)_n$, which converges weakly in $L^1(\mu)$. Recalling that D is a countable set, by a diagonal argument on $D \times \mathbb{N}$, one shows that there exists a subsequence (f_{n_k}) such that, for every $x \in D$, $(\langle x, f_{n_k}(.)\rangle)$ converges weakly to some function h_x in $L^1(\mu)$.

For every fixed $A \in \mathcal{A}$, the sequence $(\phi_{n_k}(A)) \subset E^*$ is bounded in E^* (by (9)) and $\lim_k \langle x, \phi_{n_k}(A)\rangle = \int_A h_x(a) d\mu(a)$ for every $x \in D$, a dense subset in E (for the norm topology). Thus, one deduces (see for example [30, Theorem 10 p. 125]) that the sequence $(\phi_{n_k}(A))$ converges weakly* to some element $\phi(A) \in E^*$. □

We now come back to the proof of the Theorem 3. Let $\phi : \mathcal{A} \to E^*$ be given by Claim 1. We first notice that ϕ is a vector measure, from the Vitali-Hahn-Saks theorem [18, Theorem 2, p. 158], since from above we have

$$\langle x, \phi(A)\rangle = \lim_k \int_A \langle x, f_{n_k}(a)\rangle d\mu(a).$$

In particular,

$$\forall A \in \mathcal{A},\ \hat{T}(\chi_A) = \phi(A) = \int_\Omega \chi_A(a) f(a) d\mu(a) = \int_A f(a) d\mu(a).$$

The converse implication comes from the property of the integral in $\mathbb{R}$.

Furthermore ϕ is bounded since from (9) one has

$$\forall A \in \mathcal{A},\ \|\phi(A)\| \leq \int_A g(a)d\mu(a) \leq \int_\Omega g(a)d\mu(a).$$

This above inequality also shows that ϕ is absolutely continuous with respect to μ. Hence, from the extension of the Radon–Nikodym theorem to vector measures (Theorem 4), there is a Gelfand integrable function, $f : \Omega \to E^*$, such that $\phi(A) = \int_A f(a)d\mu(a)$ for every $A \in \mathcal{A}$. Recalling that, from Claim 1,

$$\phi(A) = \lim_k \phi_{n_k}(A) = \lim_k \int_A f_{n_k}(a)d\mu(a),$$

we deduce that the sequence (f_{n_k}) converges weakly to the Gelfand integrable mapping f. □

3.2. Proof of the convex part (a) of Fatou's lemma

By Theorem 3, the sequence (f_n) is sequentially weakly compact, that is, there exists a subsequence (f_{n_k}), which converges weakly to some Gelfand integrable mapping $f : \Omega \to E^*$. Since $\lim_n \int_\Omega f_n(a)d\mu(a)$ exists from the assumption of Theorem 1, we get

$$\lim_n \int_\Omega f_n(a)d\mu(a) = \lim_k \int_\Omega f_{n_k}(a)d\mu(a) = \int_\Omega f(a)d\mu(a).$$

The end of the proof is then a consequence of Proposition 1, proven below, which implies that

$$\text{for a.e. } a \in A, \quad f(a) \in \text{cl co Ls}_k\{f_{n_k}(a)\} \subset \text{cl co Ls}_n\{f_n(a)\}.$$

The proof will thus be complete if we show the following proposition.

PROPOSITION 1. *Let (f_n) be an integrably bounded sequence of Gelfand integrable mappings from Ω to E^*, which converges weakly to a Gelfand integrable mapping $f : \Omega \to E^*$. Then, we have*

$$\textit{for a.e. } a \in A, \quad f(a) \in \text{cl co Ls}_n\{f_n(a)\}.$$

Proof of Proposition 1. By contradiction. Suppose on the contrary that there exists $A \in \mathcal{A}$, $\mu(A) > 0$, such that, for every $a \in A$, one has $f(a) \notin C(a) := \text{cl co Ls}_n\{f_n(a)\}$. Since E is separable, let D be a countable dense subset of E (for the norm topology).

CLAIM 2. *For every $a \in A$, there exists $x \in D$ such that*

$$\langle x, f(a)\rangle > \sup\{\langle x, c\rangle \mid c \in C(a)\}. \tag{10}$$

Proof of Claim 2. Indeed, by the Hahn-Banach theorem, there is x in the topological dual of E^* (endowed with the weak* topology $\sigma(E^*, E)$), which is equal to E (see, for example [30, Corollary, p. 143]), such that

$$\langle x, f(a)\rangle > \sup\{\langle x, c\rangle \mid c \in C(a)\}.$$

But x is the limit (for the norm topology) of some sequence (x^k) in D and we now show that for k large enough, x^k satisfies also the separation inequality (10). Indeed, if it is not true, for every k, there exists $c^k \in C(a)$ such that

$$\langle x^k, c^k\rangle \geq \langle x^k, f(a)\rangle - \tfrac{1}{k}.$$

Since the sequence (f_n) is integrably bounded, the set $C(a)$ is bounded (and closed), hence weak-star compact by the Alaoglu–Bourbaki theorem. Without any loss of generality, we can suppose that the sequence $(c^k) \subset C(a)$, converges to some element $\bar{c} \in C(a)$ for the weak-star topology. Passing to the limit in the above inequality, we get

$$\sup\{\langle x, c\rangle \mid c \in C(a)\} \geq \langle x, \bar{c}\rangle \geq \langle x, f(a)\rangle,$$

which contradicts the above separation inequality. □

We let, for $(x, n) \in D \times \mathbb{N}$,

$$A_{x,n} := \{a \in \Omega \mid \langle x, f(a)\rangle > \sup\{\langle x, f_m(a)\rangle \mid m \geq n\}\},$$

which is clearly a measurable set, since the mappings f and f_m are Gelfand measurable.

CLAIM 3. *There exists $(x, n) \in D \times \mathbb{N}$, such that $\mu(A_{x,n}) > 0$.*

Proof of Claim 3. Since $\mu(A) > 0$ and D is countable it is sufficient to show that $A \subset \cup_{(x,n)\in D\times\mathbb{N}} A_{x,n}$. Indeed, if it is not true, there exists a in A, such that for every $x \in D$ there exists a subsequence (m_n) satisfying

$$\forall n \in \mathbb{N}, \langle x, f(a)\rangle - \tfrac{1}{n} \leq \langle x, f_{m_n}(a)\rangle.$$

Since the sequence (f_n) is integrably bounded, from the Alaoglu–Bourbaki theorem, without any loss of generality, we can suppose that the sequence $(f_{m_n}(a))_n$ converges to some element $\bar{c} \in E^*$. Clearly, $\bar{c}$ belongs to $C(a)$ and from above we get

$$\langle x, f(a)\rangle \leq \langle x, \bar{c}\rangle \leq \sup\{\langle x, c\rangle \mid c \in C(a)\}.$$

From Claim 2, we can choose $x \in D$ satisfying the separation inequality (10), which contradicts the above inequality. □

We now come back to the proof of Proposition 1. We take (x, n) in $D \times \mathbb{N}$, and $A_{x,n}$ as in Claim 3 and we deduce that

$$\begin{aligned}\int_{A_{x,n}} \langle x, f(a)\rangle d\mu(a) &> \int_{A_{x,n}} \sup\{\langle x, f_m(a)\rangle \mid m \geq n\} d\mu(a)\\ &\geq \int_{A_{x,n}} \langle x, f_k(a)\rangle d\mu(a) \text{ for every } k \geq n.\end{aligned}$$

Taking the limit, when $k \to \infty$ we get

$$\int_{A_{x,n}} \langle x, f(a)\rangle d\mu(a) > \text{limsup}_k \int_{A_{x,n}} \langle x, f_k(a)\rangle d\mu(a),$$

which contradicts that the sequence (f_k) converges weakly to f. □

3.3. Proof of Part (b) of Fatou's lemma

We first recall some definitions. Let Φ be a correspondence from Ω to E^*, the Gelfand integral of Φ over Ω is defined to be the set

$$\int_\Omega \Phi(a) d\mu(a) := \left\{ \int_\Omega f(a) d\mu(a) \,\middle|\, \begin{array}{l} f \in L^1_G(\Omega, \mathcal{A}, \mu; E^*), \\ f(a) \in \Phi(a) \text{ a.e. } a \in \Omega \end{array} \right\},$$

and the correspondence Φ is said to be integrable if $\int_\Omega \Phi(a) d\mu(a) \neq \emptyset$ (see [1, 4, 11, 14, 21]). We point out that the integral is defined for an arbitrary correspondence Φ, and thus may be empty in many cases.

Proof of Part (b). Step 1 : proof under the additional assumption that $(\Omega, \mathcal{A}, \mu)$ is a non-atomic measure space. From Part (a) of Theorem 1, there is a Gelfand integrable mapping $f : \Omega \to E^*$ such that

$$\begin{aligned}\lim_n \int_\Omega f_n(a) d\mu(a) &= \int_\Omega f(a) d\mu(a)\\ &\in \int_\Omega \text{cl co Ls}_n\{f_n(a)\} d\mu(a), \qquad (11)\end{aligned}$$

and we claim that

$$\int_\Omega \text{cl co Ls}_n\{f_n(a)\} d\mu(a) \subset \text{cl} \int_\Omega \text{Ls}_n\{f_n(a)\} d\mu(a), \qquad (12)$$

and the closure "cl" can be suppressed in the right-hand side of the inclusion (12) when E is finite dimensional.

We postpone the proof of the inclusion (12) and we end the proof of Step 1. From (11) and (12), for every neighborhood W of 0 in E^* one has

$$\lim_n \int_\Omega f_n(a) d\mu(a) \in \left[\int_\Omega \text{Ls}_n\{f_n(a)\} d\mu(a)\right] + W.$$

Consequently, there exists an integrable mapping $f_W : \Omega \to E^*$ such that, for a.e. $a \in A$, $f_W(a) \in \mathrm{Ls}_n\{f_n(a)\}$ and

$$\lim_n \int_\Omega f_n(a) d\mu(a) - \int_\Omega f_W(a) d\mu(a) \in W.$$

Furthermore, when E is finite dimensional, from (11) and (12), the function f_W satisfies

$$\lim_n \int_\Omega f_n(a) d\mu(a) - \int_\Omega f_W(a) d\mu(a) = 0.$$

This ends the proof of Step 1. □

Before giving the proof of the inclusion (12), what will be done in the next proposition, we first need to recall some definitions. A correspondence Φ, from Ω to E^*, is said to be measurable if, for every open subset U of E^* (for the weak-star topology), the following set $\Phi^-(U) := \{a \in \Omega | \Phi(a) \cap U \neq \emptyset\}$ is measurable in Ω.

If the correspondence Φ is single-valued, hence reduces to a mapping $\varphi : \Omega \to E^*$, we shall use hereafter the fact that the three following definitions of measurability are equivalent (i) the mapping φ is measurable from $(\Omega, \mathcal{A})$ to $(E^*, \mathcal{B}(E^*))$, where $\mathcal{B}(E^*)$ denotes the Borel σ-algebra on E^* (for the weak-star topology), (ii) the correspondence Φ is measurable as defined above, and (iii) the mapping φ is Gelfand measurable (see, for example [1, 11, 21]; the implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are straightforward and for the implication $(iii) \Rightarrow (i)$, see for example, [1, Theorem 17.30, p. 576]).

We shall use several times below the fact that the closed unit ball B_* of E^* is compact and metrizable (hence separable and complete) for the weak-star topology. The compactness of B_* is exactly the Banach–Alaoglu theorem and the metrizability of B_* is a consequence of the fact that E is separable (for the norm topology).

We now state the following result, which can be proved under weaker assumptions, but is sufficient for the purpose of this article, i.e., prove the above inclusion (12).

PROPOSITION 2. *Assume in addition that the measure space* $(\Omega, \mathcal{A}, \mu)$ *is nonatomic and let* Φ *be a correspondence from* Ω *to* E^* *with nonempty (weak-star) compact values, which can be written as follows:*

$$\Phi(a) = g(a)\hat{\Phi}(a) \quad \textit{for every } a \in \Omega,$$

where $\hat{\Phi}$ *is a measurable correspondence, from* Ω *to* B_* *and the function* $g : \Omega \to \mathbb{R}$ *is positive and integrable. Then one has*

$$\int_\Omega cl\ co\ \Phi(a) d\mu(a) \subset cl \int_\Omega \Phi(a) d\mu(a), \tag{13}$$

and the closure "cl" can be suppressed in the right-hand side of the inclusion (13) when E is finite dimensional.

Furthermore, the correspondence Φ defined by $\Phi(a) = Ls_n\{f_n(a)\}$ satisfies the above assumptions on Φ.

Proof. Step 1. We first show that

$$\int_\Omega \text{cl co } \Phi(a) d\mu(a) \subset \text{cl co } \int_\Omega \Phi(a) d\mu(a). \tag{14}$$

Suppose that (14) is not true, then there is $y^* \in \int_\Omega \text{cl co } \Phi(a) d\mu(a)$ such that $y^* \notin \text{cl co} \int_\Omega \Phi(a) d\mu(a)$. Thus, by the Hahn-Banach theorem, there is x in the topological dual of E^* (endowed with the weak* topology $\sigma(E^*, E)$), which is equal to E (see, for example [30, Corollary, p. 143], such that

$$\langle x, y^* \rangle > \alpha := \sup\{\langle x, y\rangle \mid y \in \text{cl co} \int_\Omega \Phi(a) d\mu(a)\}. \tag{15}$$

But $y^* = \int_\Omega f(a) d\mu(a)$, for some integrable mapping $f : \Omega \to E^*$ such that for a.e. $a \in A$, $f(a) \in \text{cl co } \Phi(a)$ and we let

$$\hat{\Psi}(a) := \hat{\Phi}(a) \cap \{y \in B_* \in |g(a)\langle x, y\rangle \geq \langle x, f(a)\rangle\} \subset B_*.$$

We claim that the correspondence $\hat{\Psi}$, from Ω to B_* is measurable and has nonempty closed values for a.e. $a \in A$. The values of $\hat{\Psi}$ are clearly closed and we now check that they are nonempty. For a.e. $a \in A$, let $\bar{y}$ be the maximum on the set $\hat{\Psi}(a)$ (which is nonempty and weak-star compact) of the (weak-star) continuous function $y \to g(a)\langle x, y\rangle$. Then one easily checks that $\bar{y}$ belongs to $\hat{\Psi}(a)$ (recalling that $f(a)$ belongs to cl co $\Phi(a)$). Finally the correspondence $\hat{\Psi}$ is measurable since it is clearly the intersection of two measurable correspondences taking their values in B_*, which we recall is (weak*) compact, separable and metrizable, and the intersection of two measurable correspondence is also measurable (see, for example, [1, Lemma 17.3, and Theorem 17.4, p. 560])).

Since the correspondence $\hat{\Psi}$ is measurable, with nonempty closed values, it admits a measurable selection $\hat{f} : \Omega \to B_*$ (see, for example, [1, Theorem 17.16, p. 568], [12, Theorem III.38, p. 85]). Moreover $\hat{f}$ is integrable on Ω (since it takes its values in B_*) and we define $\bar{f} : \Omega \to E_*$ by $\bar{f}(a) = g(a)\hat{f}(a)$. The mapping $\bar{f}$ is clearly an integrable selection of Φ and satisfies $\langle x, \bar{f}(a)\rangle \geq \langle x, f(a)\rangle$ for a.e. $a \in \Omega$, hence

$$\int_\Omega \langle x, \bar{f}(a)\rangle d\mu(a) \geq \int_\Omega \langle x, f(a)\rangle d\mu(a) = \langle x, y^* \rangle > \alpha.$$

From above and (15), we get

$$\langle x, \int_\Omega \bar{f}(a)d\mu(a)\rangle > \alpha \geq \sup\{\langle x, y\rangle \mid y \in \int_\Omega \Phi(a)d\mu(a)\},$$

a contradiction with the fact that $\bar{f}$ is an integrable selection of Φ. This ends the proof of (14). □

Step 2. We show that the set $\operatorname{cl} \int_\Omega \Phi(a)d\mu(a)$ is convex. Indeed, let y_1, y_2 be in $\operatorname{cl} \int_\Omega \Phi(a)d\mu(a)$ and let $\alpha \in [0,1]$, we want to show that $\alpha y_1 + (1-\alpha)y_2$ also belongs to $\operatorname{cl} \int_\Omega \Phi(a)d\mu(a)$, or equivalently that, for every neighborhood of zero W in E^*

$$\alpha y_1 + (1-\alpha)y_2 \in \Big[\int_\Omega \Phi(a)d\mu(a)\Big] + W. \tag{16}$$

Since y_1 and y_2 belong to $\operatorname{cl} \int_\Omega \Phi(a)d\mu(a)$, there exist two Gelfand integrable mappings $f_i : \Omega \to E^*$ $(i = 1, 2)$ such that $f_i(a) \in \Phi(a)$ for a.e. $a \in \Omega$ and $y_i \in \int_\Omega f_i(a)d\mu(a) + \frac{1}{2}W$. To prove (16) it thus suffices to show the existence of an integrable selection $f : \Omega \to E^*$ of the correspondence Φ satisfying

$$\alpha \int_\Omega f_1(a)d\mu(a) + (1-\alpha)\int_\Omega f_2(a)d\mu(a) \in \int_\Omega f(a)d\mu(a) + \tfrac{1}{2}W. \tag{17}$$

We recall that for every w^*-neighborhood of zero W in E^* there exists a finite family $X = \{x_1, x_2, \ldots, x_n\}$ of E such that

$$\pi^{-1}(0) \subset \{y \in E^* \mid \langle x_i, y\rangle < 1 \quad i = 1 \ldots n\} \subset \tfrac{1}{2}W,$$

where $\pi : E^* \to \mathbb{R}^n$ is the mapping defined by $\pi(y) = (\langle x_i, y\rangle)_i$.

We define the vector-valued measure $\lambda : \mathcal{A} \to \mathbb{R}^n \times \mathbb{R}^n$ by

$$\forall A \in \mathcal{A} \quad \lambda(A) := \left(\int_A \pi(f_1(a))d\mu(a), \int_A \pi(f_2(a))d\mu(a)\right).$$

From Lyapunov's theorem, the range of the vector-measure λ is convex. Noticing that

$$\lambda(\emptyset) = (0,0) \quad \text{and} \quad \lambda(\Omega) = \left(\int_\Omega \pi(f_1(a))d\mu(a), \int_\Omega \pi(f_2(a))d\mu(a)\right)$$

we deduce the existence of $A \in \mathcal{A}$ such that, for $i = 1, 2$

$$\alpha \int_\Omega \pi(f_i(a))d\mu(a) + (1-\alpha)0 = \int_A \pi(f_i(a))d\mu(a).$$

We now define the mapping $f : \Omega \to E^*$ by $f = \chi_A f_1 + \chi_{\Omega\setminus A} f_2$. Then f is clearly an integrable selection of the correspondence Φ and satisfies

$$\pi\left(\alpha\int_\Omega f_1(a)d\mu(a)+(1-\alpha)\int_\Omega f_2(a)d\mu(a)-\int_\Omega f(a)d\mu(a)\right)=$$
$$\int_\Omega[\alpha\pi(f_1(a))+(1-\alpha)\pi(f_2(a))-\pi(f(a))]d\mu(a)=0.$$

Consequently

$$\alpha\int_\Omega f_1(a)d\mu(a)+(1-\alpha)\int_\Omega f_2(a)d\mu(a)-\int_\Omega f(a)d\mu(a)\in\pi^{-1}(0)\subset\tfrac{1}{2}W,$$

which ends the proof of (17) and the proof of Step 2. □

Step 3. From Step 1 and Step 2 we deduce that the inclusion (13) holds. Moreover the closure "cl" can be suppressed in the right-hand side of the inclusion (13) when E is finite dimensional. Indeed in this case the set $\int_\Omega \Phi(a)d\mu(a)$ is compact (see, for example, [21, Proposition 7, p. 73]).
□

Step 4. We now show that the correspondence Φ from Ω to E^* defined by $\Phi(a) := \mathrm{Ls}_n\{f_n(a)\}$ satisfies the assumptions of Proposition 2 (see also [10], [20] for similar proofs). Since the sequence (f_n) is integrably bounded, there exists an integrable function $g : \Omega \to \mathbb{R}$ such that, for a.e. $a \in \Omega$, $\sup_n \|f_n(a)\| \le g(a)$, and we can additionally assume that the function g is strictly positive. We define the mappings $\hat{f}_n : \Omega \to B_*$ by $\hat{f}_n(a) = f_n(a)/g(a)$ and we clearly have

$$\Phi(a) = g(a)\hat{\Phi}(a), \text{ where } \hat{\Phi}(a) := \mathrm{Ls}_n\{\hat{f}_n(a)\} \subset B_*.$$

Since the mappings $\hat{f}_n$ take their values in B_*, which is metrizable, we deduce that, for a.e. $a \in \Omega$ one has

$$\hat{\Phi}(a) := \mathrm{Ls}_n\{\hat{f}_n(a)\} = \cap_{n\in\mathbb{N}}\ \mathrm{cl}\ [\cup_{p\ge n}\{\hat{f}_p(a)\}] \subset B_*.$$

We end the proof by showing that the correspondence $\hat{\Phi}$ is measurable. Indeed, in view of the above equality the correspondence $\hat{\Phi}$ is constructed as follows. We start with the (single-valued) correspondence $a \to \{\hat{f}_p(a)\}$, which is measurable since each mapping $\hat{f}_p$ is Gelfand measurable (see for example, [1, Theorem 17.30, p. 576]). Then $\hat{\Phi}$ is obtained by taking successively the following operations on correspondences: countable union, closure and countable intersection, and each operation is known to preserve the measurability of the correspondences, when they take their values in a separable metrizable space, which is the case with B_* (see, for example, [1, Lemma 17.3, and Theorem 17.4, p. 560]). □

We now come back to the proof of Fatou's lemma.

Step 2 : proof of Part (b) *in the general case, i.e., without assuming anymore that* $(\Omega, \mathcal{A}, \mu)$ *is non-atomic.* We shall use the classical result that allows to partition the set Ω into a non atomic part, denoted $\Omega^{na} \in \mathcal{A}$, and a purely atomic part, denoted $\Omega^{pa} \in \mathcal{A}$. Furthermore the set Ω^{pa} can be written as the disjoint union of at most countably many measurable atoms $(A_i)_{i \in I}$ $(I \subset N)$. Hence, for every $i \in I$ and every $n \in \mathbb{N}$, the Gelfand measurable function $f_n : \Omega \to E^*$ takes a constant value $f_n^i \in E^*$ for a.e. $a \in A_i$. Since the sequence (f_n) is integrably bounded, the set $\{f_n^i | n \in \mathbb{N},\ i \in I\}$ is bounded, and thus remains in a weak-star compact subset of E^* by the Alaoglu–Bourbaki theorem. Consequently, by a diagonal extraction argument, there exists a subsequence (n_k) such that, for every $i \in I$, the sequence $(f_{n_k}^i)_{k \in \mathbb{N}}$ converges weakly* to some element $\bar{f}^i \in E^*$. We let $f^{pa} \colon \Omega^{pa} \to E^*$ be defined by $f^{pa}(a) = \bar{f}^i$ if $a \in A^i$, and we have shown that

$$\text{for a.e. } a \in \Omega^{pa},\ f^{pa}(a) \in \mathrm{Ls}_n\{f_n(a)\}.$$

We now show that

$$\lim_k \int_{\Omega^{pa}} f_{n_k}(a) d\mu(a) = \int_{\Omega^{pa}} f^{pa}(a) d\mu(a).$$

This is clearly a consequence of the Lebesgue dominated convergence theorem, applied for every fixed $x \in E$, to the sequence of integrably bounded real-valued functions $(\langle x, f_{n_k}(.)\rangle)_{k \in \mathbb{N}}$ (since the sequence (f_{n_k}) is also integrably bounded on Ω^{pa}).

We now consider the non atomic part Ω^{na} and we first remark that $\lim_k \int_{\Omega^{na}} f_{n_k}(a) d\mu(a)$ exists and

$$\lim_k \int_{\Omega^{na}} f_{n_k}(a) d\mu(a) = \lim_k \int_{\Omega} f_{n_k}(a) d\mu(a) - \lim_k \int_{\Omega^{pa}} f_{n_k}(a) d\mu(a),$$

since the two limits in the right-hand side of the equation exist (the first one from the assumption made in Theorem 1, and the second one from above).

We can now apply to the sequence $(f_{n_k})_{k \in \mathbb{N}}$ and to the non atomic part Ω^{na}, the version of Fatou's lemma proved in the first step. Thus, for every neighborhood W of 0 in E^*, there is a Gelfand integrable mapping $f^{na} : \Omega^{na} \to E^*$ such that

$$\text{for a.e. } a \in \Omega^{na},\ f^{na}(a) \in \mathrm{Ls}_k\{f_{n_k}(a)\} \subset \mathrm{Ls}_n\{f_n(a)\},$$

$$\lim_k \int_{\Omega^{na}} f_{n_k}(a) d\mu(a) - \int_{\Omega^{na}} f^{na}(a) d\mu(a) \in W.$$

Furthermore, when E is finite dimensional, one has

$$\lim_k \int_{\Omega^{na}} f_{n_k}(a)d\mu(a) - \int_{\Omega^{na}} f^{na}(a)d\mu(a) = 0.$$

We now define the mapping $f_W : \Omega \to E^*$ by $f_W(a) = f^{pa}(a)$ if $a \in \Omega^{pa}$ and $f_W(a) = f^{na}(a)$ if $a \in \Omega^{na}$, and one checks that the mapping f_W satisfies the condition of Theorem 1. □

3.4. Proof of Fatou's lemma with Bochner integral

We give the proof under the additional assumption that $(\Omega, \mathcal{A}, \mu)$ is a non-atomic measure space. The general case is then an adaptation of the proof given in the previous section (see the proof of Step 2). Let (f_n) be a sequence of Bochner integrable mappings from Ω to E^* satisfying the assumptions of Theorem 2, we want to show that, for every $\epsilon > 0$, there exists a Bochner integrable mapping $f_\epsilon : \Omega \to E^*$ such that:

$$\left\| \int_\Omega f_\epsilon(a)d\mu(a) - \lim_n \int_\Omega f_n(a)d\mu(a) \right\| \leq \epsilon,$$
$$\text{and, for a.e. } a \in \Omega,\ f_\epsilon(a) \in \mathrm{Ls}_n\{f_n(a)\}.$$

This is equivalent to show that

$$\lim_n \int_\Omega f_n(a)d\mu(a) \in \text{s-cl} \int_\Omega^B \mathrm{Ls}_n\{f_n(a)\}d\mu(a),$$

where "s-cl" denotes the closure for the norm-topology of E^* (recalling that "cl" denotes the closure for the weak-star topology) and the Bochner integral of a correspondence Φ from Ω to E^* is defined to be the set[6]

$$\int_\Omega^B \Phi(a)d\mu(a) := \left\{ \int_\Omega f(a)d\mu(a) \;\middle|\; \begin{array}{l} f : \Omega \to E^* \text{ is Bochner integrable,} \\ \text{and } f(a) \in \Phi(a) \text{ for a.e. } a \in \Omega \end{array} \right\}.$$

But, from Part (b) of Theorem 1, we deduce that

$$\lim_n \int_\Omega f_n(a)d\mu(a) \in \mathrm{cl} \int_\Omega \mathrm{Ls}_n\{f_n(a)\}d\mu(a),$$

hence the proof of Theorem 2 reduces to show the following claim:

[6] Above and below, we denote identically the Bochner and the Gelfand integrals of a Bochner integrable *mapping* $f : \Omega \to E^*$ since both integrals coincide. We only use different notations for the Bochner and the Gelfand integrals of a *correspondence* since the latter may be greater in general (but will be shown to coincide in the case we consider here).

$$\text{cl} \int_\Omega \text{Ls}_n\{f_n(a)\} d\mu(a) \subset \text{s-cl} \int_\Omega^B \text{Ls}_n\{f_n(a)\} d\mu(a). \tag{18}$$

Indeed, to prove (18), we first deduce from Mazur's theorem (see, for example, [30, Theorem 2, p. 120], recalling that E^* is reflexive, hence the weak-star topology on E^* coincides with the weak topology on E^*) that, for every $L \subset E^*$ one has cl co $L =$ s-cl co L, hence

$$\text{cl} \int_\Omega \text{Ls}_n\{f_n(a)\} d\mu(a) \subset \text{s-cl co} \int_\Omega \text{Ls}_n\{f_n(a)\} d\mu(a). \tag{19}$$

We now check that the notions of Gelfand and Bochner integrals coincide, in fact, we only need to show that

$$\int_\Omega \text{Ls}_n\{f_n(a)\} d\mu(a) \subset \int_\Omega^B \text{Ls}_n\{f_n(a)\} d\mu(a). \tag{20}$$

Indeed, let $y \in \int_\Omega \text{Ls}_n\{f_n(a)\} d\mu(a)$, i.e., $y = \int_\Omega f(a) d\mu(a)$ for some Gelfand integrable mapping $f : \Omega \to E^*$ such that $f(a) \in \text{Ls}_n\{f_n(a)\}$ for a.e. $a \in \Omega$. Since E is separable and reflexive, the dual space E^* is also separable (for the norm topology). Hence, by Pettis' theorem (see, for example [30, p. 131]), the mapping $f : \Omega \to E^*$, which is Gelfand measurable, is strongly measurable. Moreover, since the sequence (f_n) is integrably bounded (by the real-valued integrable function $g : \Omega \to \mathbb{R}$) we deduce that, for a.e. $a \in \Omega$, $\|f(a)\| \le g(a)$, thus the function $a \mapsto \|f(a)\|$ is integrable. This property, together with the strong measurability of f implies that f is Bochner integrable (see, for example, [30, Theorem 1, p. 133]). Hence $y \in \int_\Omega^B \text{Ls}_n\{f_n(a)\} d\mu(a)$, which ends the proof of (20).

We now can end the proof of Claim (18). Indeed this is a consequence of (19), (20) and the fact that the set s-cl co $\int_\Omega \text{Ls}_n\{f_n(a)\} d\mu(a)$ is convex, recalling that the measure space $(\Omega, \mathcal{A}, \mu)$ is assumed to be nonatomic (see, for example, [19, Theorem 4.2]). □

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