Economic Equilibrium: Optimality and Price Decentralization

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Abstract. Mathematical economics has a long history and covers many interdisciplinary areas between mathematics and economics. At its center lies the theory of market equilibrium. The purpose of this expository article is to introduce mathematicians to price decentralization in general equilibrium theory. In particular, it concentrates on the role of positivity in the theory of convex economic analysis and the role of normal cones in the theory of non-convex economies.

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1. A Historical Survey

General equilibrium theory describes the equilibrium and disequilibrium arising from the interaction of all economic agents in all markets. The basic abstractions in the model of general competitive equilibrium are the notions of commodities and prices. Commodities define the universe of discourse within which the constraints, motivations, and choices of consumers and producers are set. Consumers and producers act independently and respond to prices. At equilibrium, a linear price system summarizes the information concerning relative scarcities

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and locally approximates the possibly non-linear primitive data of the economy.

Advances in the theory of general equilibrium have gone hand-in-hand with the study of the existence of at least one equilibrium price system. This is not surprising since the existence problem was far more involved than what many economists had anticipated in the past. With complexity came the need for rigor and rigor lead to a better understanding of not only the existence problem but also the model as a whole.

The purpose of this section is to informally and summarily trace the evolution of the general equilibrium model from Léon Walras' system of production and exchange equations to the 'state of the art' model with infinitely many commodities and a finite number of consumers and producers.

1.1. Finite Number of Commodities

Léon Walras: The classical reference to the theory of general equilibrium is Léon Walras' *Elements of Pure Economics*, which was published in four successive editions between 1874 and 1900 and in a further 'definitive' edition, published sixteen years after Walras' death.[1] Walras in *Elements* (§162–163) concedes to Gossen the priority of discovering the principle of 'maximization of utility' and to Jevons the priority of discovering the 'equations of exchange.' Walras, however, is the only one out of the three to deal with the case of more than two commodities and two consumers.[2] His own contribution, he remarks, is "the general case in which any number of individuals enter into mutual exchange relations... on the supposition that any number of commodities are being exchanged for one another." [114, §163, p. 206].

The first problem that Walras considers by means of "marginal utility" (first order derivatives), or to use Walras' term *rareté*, was the problem of barter. Walras derives in Lessons 11–15 of *Elements* a system of equations, which he calls the 'equations of exchange.' A solution to this system of equations is an equilibrium for an exchange economy, even in our modern understanding of the term. A solution to Walras' 'production equations,' in Lessons 17–22, on the other hand, is very different from the modern notion of production equilibrium. Walras does not include a fixed number of profit maximizing producers. Consumers trade production inputs and transform them into consumable goods, which they either trade or consume themselves. In this way production decisions are primarily motivated by utility maximization. One can

[1] We cite William Jaffé's English translation of the 'Definitive Edition' [114].

[2] See "Walras on Gossen" in [113] and Schumpeter [103, pp. 103–126].

think of Walras' production model as one in which each consumer holds a one hundred percent share in every available technology.

Walras goes through a great deal of effort to show that each system of equations reduces to a system with an equal number of variables as equations. This appears to have been the usual practice in the physical sciences of the late nineteenth century. The following is a quote from a review of Pareto's *Manuel d'Économie Politique* from the June 1912 issue of the *Bulletin of the American Mathematical Society*:

> *"...it should be remembered that not so very long ago the method of counting constants was widely used in pure mathematics... . Moreover, in a physical science the question of rigor is very different from that in mathematics; to be ultra rigorous mathematically may be infra-rigorous physically. To throw out Gibb's phase because its proof, being essentially a count of constants, is no proof at all, would be equally good mathematics and equally bad physics."* [114, p. 511].

Walras however fully recognizes that even his two commodity exchange model may have *no* equilibrium (§64) or *multiple* equilibria (§65). Walras also recognizes that there can be a *stable* equilibrium (§66), an *unstable* equilibrium (§67), and *multiple* equilibria some of which are stable and some that are not (§68). However, in §156 he makes the interesting (but probably wrong) assertion that when "the number of commodities is very large" multiple equilibria "are, in general, not possible."

Abraham Wald: The first rigorous results on the existence of a solution to Walras' 'production equations' and 'exchange equations' is due to a series of papers, appearing between 1933 and 1936, by Abraham Wald.[3] Wald recognizes that a "theorem on the solubility of the equations under consideration can only be proven to follow from the assumptions by means of difficult mathematical analysis." [112, p. 403].

Most historical accounts concentrate on the unsatisfactory aspects of Wald's production theorem, while ignoring the tremendous breakthrough on the exchange front. Wald's assumptions for the consumption case are remarkably weak. He assumes the following:

1. Each consumer holds a non-negative amount of each commodity.

2. Each consumer holds a strictly positive amount of some commodity.

3. Each commodity is held in a strictly positive amount by some consumer.

[3] See Wald [112] for a report on his work.

4. For each consumer the marginal utility of a good is independent of the amount held of other goods, and is strictly monotonically decreasing with the amount held of the good.

Condition (4) of Wald means that the utility functions are separable and is identical to Walras' assumptions on the *rareté*. Thus, in so far as the Walrasian 'exchange equations' are concerned, Wald solved the existence problem.

Concerning Wald's theorem for the 'production equations' it suffices to say that he describes the consumption sector by inverted demand functions, where prices are functions of quantity. Wald's system of equations is in fact the Cassel system of equations. Though Walras expressed demand as a function of prices, Cassel [41] uses the inverses of the demand functions and assumes that these inverses are themselves functions. He assumes, among other things, that these inverted demand functions are continuous and satisfy a condition almost identical to the well known Samuelson's weak axiom of revealed preferences; see [83].

The Arrow–Debreu–McKenzie Model: The problem of existence of equilibrium stayed uninvestigated for about two decades after the publication of Wald's work. In the early fifties, stimulated by the advances in linear programming, activity analysis, and game theory, "it was perceived independently by a number of scholars that existence theorems of greater simplicity and generality than Wald's were now possible." [18, p.11]. By 1956 numerous equilibrium existence results were independently obtained. Some of these are in McKenzie [84], Arrow and Debreu [17], Gale [62], and Nikaido [93]. One of the most general finite dimensional existence result is that of Debreu [54].

Two characteristics of these results are worth mentioning. First, the problem of existence was no longer perceived as that of solving a system of equations but was reformulated as a problem of showing that the simultaneous maximization of individual goals under independent constraints can be carried out. These saw a departure from the calculus based marginal utilitarian analysis to functional analytical techniques; and according to Debreu [53, p. x] new influences "freed mathematical economics from its tradition of differential calculus and compromises with logic." Second, all these results were obtained by applying a fixed point argument; some variant of Brouwer's fixed point theorem. It is often stated that Léon Walras could not have proven the existence of equilibrium since he did not have available Brouwer's fixed point theorem—proven in 1912. We note, however, that Léon Walras was in correspondence with the famous mathematician Henri Poincaré who in 1883–1884 announced the following result (see [36, p.51]):

(**Poincaré**) *Let* $\xi_1, \xi_2, \ldots, \xi_n$ *be* n *continuous functions of* n *variables* $x_1, x_2, \ldots, x_n$. *The variable* x_i *is subject to vary between limits* $+a_i$ *and* $-a_i$. *Let us suppose that for* $x_i = +a_i$, ξ_i *is constantly positive, and for* $x_i = -a_i$, ξ_i *is constantly negative,; I say that there exists a system of values of* x *for which all* ξ*'s vanish.*

Poincaré's theorem is now known as Miranda's fixed point theorem and is equivalent to Brouwer's fixed point theorem (see [88]) and can be used to prove the existence of equilibrium.

Apart from this technical revolution a new notion of commodities had emerged. Commodities were now contracts promising the delivery of a good or service g, in a specific location l, on a specific date t, and contingent on a series of uncertain events occurring up to date t (see [53, Chapter 7]). Endowed with this notion of commodities, the Walrasian general equilibrium model became known as the Arrow–Debreu–McKenzie general equilibrium model.

The archetype general equilibrium model with a finite number of commodities is that of Arrow and Debreu [17]. The standard reference to this model is Debreu's classic monograph *Theory of Value, an Axiomatic Study of Economic Equilibrium*, which was published in 1959. A central theme of Debreu is the axiomatic method in which the mathematical theory "is logically entirely disconnected from its interpretation." [53, p. viii].

The contribution that such a dichotomy made to economics cannot be overestimated. The axiomatic method provided for the rapid advancement of economics: Problems that were of unimaginable complexity could now be clearly formulated and solved. With the axiomatic method, economists could now focus on the most minute features of a suitably formulated theory. The Arrow–Debreu–McKenzie 'private ownership economy' is formulated as follows:

Commodities and prices. There is a finite number ℓ of commodities. Both *commodity bundles* and *price systems* are points in $\mathbb{R}^\ell$. The *value* of a commodity bundle $x \in \mathbb{R}^\ell$ relative to a price system $p \in \mathbb{R}^\ell$ is $p \cdot x$, the scalar product in $\mathbb{R}^\ell$.

Producers. There is a nonempty finite set J of producers. Each producer $j \in J$ is described by a *production set* Y_j, a subset of the commodity space $\mathbb{R}^\ell$ which describes the technological possibilities of the producer. Points in Y_j are called *production plans.* The positive coordinates of a production plan are the *producer's outputs* while the negative coordinates are the *producer's inputs.* A producer's *profit* relative to a production plan and a price system is the value of the production plan relative to the price system. Given a price system producers choose a production plan that maximizes profit over their production sets.

Consumers. There is a nonempty finite set I of consumers. Each consumer $i \in I$ is described by a *consumption set* X_i (a subset of the commodity space $\mathbb{R}^\ell$), a *preference relation* on X_i, a commodity bundle $\omega_i \in X_i$ (called the *initial endowment*), and a *share* $\theta_{ij} \in [0,1]$ of the j^{th} producer's profits. Points in X_i are called *consumption bundles*. The positive coordinates of a consumption bundle are the consumer's *inputs* while the negative coordinates are the consumer's *outputs*. Given a price system and a production plan for each producer, a consumer's *budget* set is the set of all consumption bundles whose values are not greater than the value of the initial endowment plus his share of total profits. Given a price system and a production plan for each producer, a consumer chooses a consumption bundle that maximizes her preference relation relative to her budget set.

Equilibrium. An *attainable* (or a *feasible*) *allocation* is a set consisting of a production plan for each producer and a consumption bundle for each consumer such that the sum of all consumption bundles is equal to the sum of all production plans plus the sum of the initial endowments. An *equilibrium price system* is a price system p such that consumer and producer choices relative to p is a feasible allocation.

Extensions and generalizations: The literature on equilibrium with a finite number of commodities is enormous. The assumptions used have been considerably weakened. For instance, McKenzie [86] showed that the assumption of irreversibility of total production is superfluous; McKenzie [85] and Shafer and Sonnenschein [106] allowed for interdependent preferences; McKenzie [85], Bergstrom [23] and many others weakened the assumption of free disposability; McKenzie [87] adapted the excess demand approach to the case of intransitive and incomplete preferences.

In their remarkable papers Debreu and Scarf [56] and Aumann [19] characterize market equilibrium with many traders in terms of a cooperative notion of equilibrium. Aumann considers the case of infinitely many consumers and Debreu and Scarf consider infinite replications of finite economies.

Another approach for proving the existence of equilibrium with a finite number of commodities is that of Negishi [92], which was extended by Takayama and El-Hodiri [108] and Arrow and Hahn [18].

This approach involves finding efficient feasible allocations and price systems that support them and then applying a fixed point argument on the space of utilities.

One of the most important generalizations of Debreu's assumptions is the work on intransitive and incomplete preferences. Empirical evidence suggests that consumers sometimes display cyclical preferences (e.g., Sonnenschein [107]). Also, a bounded rationality approach to consumer behavior suggests that preferences may be incomplete; there may exist two commodity bundles that cannot be compared in terms of preferences. The first satisfactory solution for the existence of equilibrium with intransitive preferences was provided by Mas-Colell in [80]. Mas-Colell showed that both the assumption of transitivity and the assumption of completeness are superfluous. Mas-Colell's proof however was long and very difficult. A shorter more appealing proof was later provided in Gale and Mas-Colell [63], which consists of three steps. First, it defines an abstract game using the data of the economy. Second, it shows that this abstract game has an equilibrium. Third, it proves that an equilibrium for the abstract game is also an equilibrium for the economy. Of course, this method of proof was not new. For example, Arrow and Debreu [17] use a similar approach in their proof of the existence of equilibrium. However, what was new was the specific form of the abstract game. Further generalizations of the Gale–Mas-Colell result and new proofs were provided in many papers (e.g., Shafer and Sonnenschein [104, 105, 106]).

The problem of uniqueness of equilibrium has been studied using techniques from global analysis in a list of papers following Debreu's seminal contribution [55]. The standard reference on the approximation of equilibria is Scarf [102]. Recently, there have been important advances in the theory of general equilibrium with incomplete markets; see for instance Geanakoplos [64]. Two surprising things have emerged from this research. First, that Brouwer's fixed point theorem is not powerful enough to prove the existence of equilibrium with incomplete assets markets. Second, that an equilibrium for such a market need not be constraint efficient.

1.2. Infinite number of commodities

The definition of a commodity in the Arrow–Debreu–McKenzie model lead inevitably to the need for considering a model with infinitely many commodities. Such a situation arises if one wants to consider economies extending over an infinite horizon, or where time or location are taken as continuous variables. It is also needed in the case of uncertainty with infinitely many states of nature.

However, the need for considering infinitely many commodities was perceived even before the Arrow–Debreu–McKenzie model was developed. For example, Schumpeter [103, p. 900] in his discussion of Frank Knight's criticism of the theory of factors of production remarks that "it would hardly be easy to eliminate entirely the idea of factors. For... Professor Knight... admits an indefinite variety of factors within which there is no economically significant difference," and that "[in] strict logic, the number of these factors would be infinite, for conceptually they form a continuum."

Among the first articles to consider the case of infinitely many commodities within a formal general equilibrium framework was Debreu's 1954 paper [51] *Valuation Equilibrium and Pareto Optimum*. Debreu's model is a strict mathematical generalization of the model with a finite number of commodities. He substitutes for the commodity space $\mathbb{R}^\ell$ an infinite dimensional topological vector space, while generalizing the concept of a price system to that of a continuous linear functional on the commodity space. The same approach to the theory of value with infinitely many commodities was taken by Hurwicz [68], who studied, among other things, linear programming in models with infinitely many commodities.

Debreu shows that a Pareto optimal allocation for a production economy can be supported by a price system. He makes, however, a strong assumption on the production set: he assumes that the production set has a non-empty interior. He then notes that this non-interiority assumption is satisfied if the commodity space is properly chosen and if free disposal of commodities is assumed. By 'properly chosen' Debreu means the infinite dimensional topological vector space L_∞, which is the space of essentially bounded measurable real valued functions on a measure space. The importance of L_∞ lies in the fact that its canonical positive cone has a non-empty interior.

In a paper published in 1967, Radner [97] pointed out that Debreu's notion of a price system is too general from an interpretive point of view. This is because some continuous linear functionals (the singular ones) do not maintain the traditional interpretation of a price system as a list of prices. He therefore proposes that for the commodity space L_∞ an appropriate price system would be an integrable real valued function, i.e., a function in L_1. Such functions, he argued, both retain the desired economic interpretation as well as being continuous linear functionals.

Vector lattice commodity spaces: The first result on the existence of equilibrium in economies with commodity space L_∞ and the price space is L_1 was presented by Bewley [24]. However, it became quickly apparent that Bewley's result is not easily extendable to the case of ordered vector paces without order unit; or even to an ordered Banach space whose positive cone contains no interior points.

In 1983 Aliprantis and Brown [2] proposed that the appropriate setting for infinite dimensional equilibrium theory is that of a vector lattice dual system. Subsequently, a general existence of equilibrium result was obtained by Mas-Colell [81] in 1986. His solution replaces the requirement of the existence of an order unit with two assumptions. First, that the commodity space be a locally solid vector lattice and that the price space be its topological dual. Second, that preferences satisfy a cone condition that he termed *uniform properness*; this condition is closely related to the condition of Klee [74] for the existence of supporting hyperplanes. For more on such cone conditions see [11].

During the last two decades order theoretic properties of vector spaces have become central to equilibrium analysis in economics, see for example [3, 4, 5, 8, 9, 13, 15, 20, 57, 76, 82, 95, 98, 99, 109, 110, 116].

It has become apparent that the existence of equilibrium prices in general order vector spaces can be obtained by using the famous Riesz–Kantorovich formula, where for each pair p and q of linear functionals and each $x \geq 0$ their Riesz–Kantorovich formula is defined by

$$\mathcal{R}_{p,q}(x) = \sup\{p(y) + q(z)\colon\ y, z \geq 0 \ \text{ and } \ y + z = x\}\,.$$

The above formula defines automatically a linear price if the underlying ordered vector space is a vector lattice. However, in the general case of ordered vector spaces the formula still defines non-linear functions that can be used to develop a new theory of economic equilibrium, see [10, 13]. Remarkably, if one can find two positive linear functionals on an ordered topological vector space for which the supremum exists but does not satisfy the Riesz–Kantorovich formula, then one can construct an economy that satisfies all the standard assumptions but fails to have an equilibrium supported by linear prices [89]. These economic results are closely related to the following open problem regarding linear operators between vector lattices:

- *If L and M are vector lattices with M not Dedekind complete and the supremum (least upper bound) $S \vee T$ of two operators $S, T \in \mathcal{L}^{\sim}(L, M)$ exists in $\mathcal{L}^{\sim}(L, M)$, does it then satisfy the Riesz–Kantorovich formula?*

Economic insights have provided partial solutions to this problem in the case of $M = \mathbb{R}$; for details see [12, 10].

1.3. Non-Convex Economies

All the literature that we have talked about so far deals with convex economies, i.e., convex production sets, and both convex consumption sets and convex preferences. We shall focus hereafter to the study of "non-convexities" in the production side. In the consumption side "non-convexities" can be treated by allowing the set of consumers to be infinite (say $[0,1]$ as in Aumann [19], or more generally, a positive measure space, as in Hildenbrand [66]), so that every agent is "negligible," hence giving an idealistic formalization of perfect competition. This is typically what cannot be assumed on the production side, since "non-convexities" are generally present in "big" firms.

The presence of "non-convexities" in the production sector is widely recognized in the economic literature and the failure of the competitive mechanism in such an environment has been known since Marshall [79]. Indivisibilities, increasing returns and fixed costs are the most common forms of non-convexities in production. The study of indivisibilities needs different types of mathematical techniques (mainly discrete mathematics). The two other forms of non-convexities arise in the case of public monopolies, such as railways or electricity production, for which the state intervention is often suggested. In these cases profit maximization is typically replaced by marginal cost pricing following the works of Allais [14], Hotelling [**?**, 67], Lange [75], Lerner [77], Pigou [94], for example. Non-convexities are part of a long tradition which dates back to Dupuit [61] and his work in 1844 on public utility pricing.

The treatment of these different questions has benefited from the development over the past twenty years of new mathematical techniques, known as "Non-smooth Analysis," which are discussed in the books by Clarke [43] and Rockafellar [**?**].

2. The Economic Model

2.1. The Commodity-Price Duality

One of the basic economic characteristics associated with any economic model is the *commodity-price duality*. G. Debreu [51] expressed this in terms of a dual pair $\langle L, L'\rangle$.[4] The vector space L is the *commodity space* and its vectors are called *commodity bundles* and L' is the *price space* and its vectors are called *prices*. The real number $\langle x, x'\rangle$ is the

[4] That is, L and L' are vector spaces that are related via a non-trivial bilinear function $\langle\cdot,\cdot\rangle\colon L \times L' \to \mathbb{R}$ called the *valuation* of the dual pair.

value of the bundle x at price x'. It is customary in economic theory to designate the price vectors by p, q, etc., instead of using primes. Also, the valuation $\langle x, p\rangle$ is denoted $p \cdot x$, i.e., $p \cdot x = \langle x, p\rangle$.

We shall now consider on the commodity space L a partially (ordered) structures and we need first to recall some definitions and results.

An *ordered vector space* L is a real vector space L equipped with an order relation $\geq$ (i.e., with a transitive, reflexive, and antisymmetric relation $\geq$) compatible with the algebraic structure of E in the sense that it satisfies the following two properties:

(i) if $x, y \in L$ satisfy $x \geq y$, then $x + z \geq y + z$ holds for all $z \in L$,

(ii) if $x, y \in L$ satisfy $x \geq y$, then $\lambda x \geq \lambda y$ for each $\lambda \in \mathbb{R}$ with $\lambda \geq 0$.

The notation $y \leq x$ will be used interchangeably with $x \geq y$. The relation $x > y$ (resp. $y < x$) will mean $x \geq y$ (resp. $y \leq x$) and $x \neq y$. The elements x of E with $x \geq 0$ are called *positive elements* or *positive vectors*. The set $L_+ = \{x \in L\colon\ x \geq 0\}$ is called the *positive cone* (or simply the *cone*) of E and it satisfies the following three properties:

(a) $L_+ + L_+ \subseteq L_+$, where $L_+ + L_+ = \{x + y\colon\ x, y \in L_+\}$;

(b) $\lambda L_+ \subseteq L_+$ for each $0 \leq \lambda \in \mathbb{R}$, where $\lambda L_+ = \{\lambda x\colon\ x \in L_+\}$;

(c) $L_+ \cap (-L_+) = \{0\}$, where $-L^+ = \{-x\colon\ x \in L^+\}$.

Any subset C of a real vector space L satisfying properties (a), (b), and (c) is called a *pointed convex cone* of L, or simply a *cone* hereafter. If C is a cone of L, note that the relation $x \geq y$ whenever $x - y \in C$ makes L an ordered vector space whose positive cone is precisely C.

We shall later introduce an order (lattice) structure on the commodity and price spaces. Throughout this paper we assume that L is a Hausdorff locally convex space and L' is its topological dual. Thus both L and L' are equipped with both a linear structure and a topological structure.

2.2. Consumers

There is a finite set I of consumers indexed by i. Each consumer $i \in I$ can consume vectors in the agent's *consumption set* $X_i \subseteq L$. Unless otherwise stated, we assume that $X_i = L_+$ for each consumer.

Each consumer i has a preference relation $\succeq_i$ on X_i. Preferences and utility functions are among the fundamental concepts of microeconomic theory. They provide the mathematical framework for modeling the "tastes" of the consumers. We present their mathematical background in this section. Hereafter we shall omit the index i of the consumer.

A *preference* $\succeq$ on a set X is a binary relation on X that is:

1. **reflexive**: $x \succeq x$ for all $x \in X$,

2. **transitive**: $x_1 \succeq x_2$ and $x_2 \succeq x_3$ imply $x_1 \succeq x_3$, and

3. **complete**: for each $x_1, x_2 \in X$ either $x_1 \succeq x_2$ or $x_2 \succeq x_1$.

When $x_1 \succeq x_2$ holds, we say that x_1 *is preferred or is indifferent to* x_2. When both $x_1 \succeq x_2$ and $x_2 \succeq x_1$ hold, then x_1 is *indifferent* to x_2, written $x_1 \sim x_2$. We also associate to $\succeq$ the *strict preference relation* $\succ$ which is defined by $x_1 \succ x_2$ if both $x_1 \succeq x_2$ and $x_2 \not\succeq x_1$.

A preference relation $\succeq$ on a topological space X is:

- **upper semicontinuous**, if for each $x \in X$ the *strictly worse-than-*x set $\{x' \in X : x \succ x'\}$ is open in X (for the relative topology),

- **lower semicontinuous**, if for each $x \in X$ the *strictly better-than-*x set $\{x' \in X : x' \succ x\}$ is open in X,

- **continuous**, if $\succeq$ is both upper and lower semicontinuous,

- **locally non-satiated at** x, if x belongs to the closure of the strictly better-than-x set $\{x' \in X\colon\ x' \succ x\}$.

When X is a convex subset of a vector space, we shall say that:

- a preference relation $\succeq$ on X is **convex**, if for each $x \in X$ the set $\{x' \in X\colon\ x' \succeq x\}$ is convex,

- a strict preference relation $\succ$ on X is **convex**, if for each $x \in X$ the set $\{x' \in X\colon\ x' \succ x\}$ is convex.

In case the set X is partially ordered by $\geq$,[5] then we say that a preference relation $\succeq$ on X is:

- **monotone**, if $x_1 \geq x_2$ implies $x_1 \succeq x_2$,

- **strictly monotone**, if $x_1 > x_2$ implies $x_1 \succ x_2$.

[5] To any partial order $\geq$ we define (as usual) the relation $>$ by letting $x_2 > x_1$ if $x_2 \geq x_1$ and $x_1 \neq x_2$. When $X = \mathbb{R}^\ell$, we shall consider the partial order defined by the positive orthant $\mathbb{R}^\ell_+ = \{x = (x_h) \in \mathbb{R}^\ell\colon\ x_h \geq 0 \text{ for every } h\}$.

A *utility function* for a preference relation $\succeq$ on X is a function $u: X \to \mathbb{R}$ such that for every x_1 and x_2 we have $u(x_1) \geq u(x_2)$ if and only if $x_1 \succeq x_2$. In this case we say u *represents* $\succeq$ on X. Every real-valued function $u: X \to \mathbb{R}$ gives rise automatically to a preference relation $\succeq$ defined by $x_1 \succeq x_2$ if $u(x_1) \geq u(x_2)$. Note that a utility function is quasi-concave if and only it represents a convex preference relation. For the existence of a utility function representing a given preference relation, we refer to the classical article by Debreu [52] and for more recent generalizations to [35].

When preferences are not complete or transitive we usually describe the tastes of an agent by a correspondence $P: X \twoheadrightarrow X$. The set $P(x)$ is interpreted as the *strictly better-than-x* set. We assume in this case that the correspondences are *irreflexive* in the sence that $x \notin P(x)$ for all x in X. Each preference relation $\succeq$ on X defines its own correspondences $P: X \twoheadrightarrow X$ by letting

$$P(x) = \{x' \in X:\ x' \succ x\}\,.$$

An element $a \in X$ is *maximal* for a preference correspondence $P: X \twoheadrightarrow X$ if $P(a) = \emptyset$. When a preference $\succeq$ is complete and reflexive an element a is maximal if and only if it is a *greatest element*, i.e., $a \succeq x$ for all $x \in X$. For economic applications and several results on maximal elements of preference correspondences see [1, 115].

2.3. Producers

There is a finite set J of producers indexed by j. A *producer* (also called a *firm* or a *technology of production* or even a *sector of production*) is an abstract "entity" to which some knowledge (technological possibility) is available that allows him to produce some outputs from one or more inputs.

The knowledge of the producer $j \in J$ is represented by a subset Y_j of the commodity space L, which gathers all its technologically possible production plans. The set Y_j is called the *production set* of j. A *production plan* for the j^{th} producer is a vector $y \in Y_j$.

When L is a vector lattice and $y = y^+ - y^-$ is a production plan, y^+ is the *output* of the producer and y^- comprises the *input*. Moreover, if $p \in L'$ is a price, then the *cost* of production is $p \cdot y^-$ and the *revenue* is $p \cdot y^+$. The *profit* of the producer for the plan y is $p \cdot y = p \cdot y^+ - p \cdot y^-$, that is, revenue minus cost.

Producers seek to maximize profit. That is, given a price $p \in L'$ the producer j wants to choose some $y \in Y_j$ so that:

$$p \cdot y \geq p \cdot y' \quad \text{for all} \quad y' \in Y_j\,.$$

As we shall see later, this maximizing behavior basically requires that each production set (or the total production set $Y = \sum_{j \in J} Y_j$) is convex. In the sequel, we shall consider production sets that are not convex and therefore they introduce alternative objectives for the producers.

Assumption (P) *Every production set Y_j is closed and satisfies the following properties:*

- $0 \in Y_j$ (***possibility of inaction***)
- $Y_j - L_+ \subseteq Y_j$ (***free disposal***)

The free disposal assumption has the following interpretation in the vector lattice setting: if $y \in Y_j$ is a production plan, then every plan $z \in L$ with higher input than y (i.e., $z^- \geq y^-$) and output lower than the output of y (i.e., $z^+ \leq y^+$) is a production plan (i.e., $z \in Y_j$).

When the positive cone L_+ has a nonempty interior, we point out that Assumption (P) implies that the boundary ∂Y_j of the production set Y_j coincides with the set of (weakly) *efficient production plans*, that is,

$$\partial Y_j = \{y_j \in Y_j \colon \ \nexists\, z_j \in Y_j \ \text{ such that } \ z_j \gg y_j\} \,,$$

where as usual $x \gg y$ means that $x - y \in \operatorname{int} L_+$.

2.4. The Economy

The economy has an *initial endowment* $\omega \in L_+$, and the order interval $[0, \omega] = \{x \in L : 0 \leq x \leq \omega\}$ is known as the *Edgeworth box* of the economy.

An *economy* is a list

$$\mathcal{E} = (I, J, \langle L, L' \rangle, (X_i, \succeq_i)_{i \in I}, (Y_j)_{j \in J}, \omega) \,,$$

where:

- The finite sets I and J are, respectively, the sets of consumers and producers .
- $\langle L, L' \rangle$ is the commodity-price duality. (As mentioned earlier, L is also equipped with a Hausdorff locally convex topology so that L' is its topological dual.)
- Each consumer $i \in I$ has a consumption set $X_i \subseteq L$, an initial endowment $\omega_i \in X_i$, and a preference relation $\succeq_i$ on X_i.
- Each producer $j \in J$ has a production set $Y_j \subseteq L$.

- $\omega \in L$ is the total initial endowment of the economy.

An *exchange economy* is an economy without production. That is, it is an economy where there are no producers; or where all producers are inactive, i.e., $Y_j = \{0\}$ for every producer j.

An *allocation* for the economy $\mathcal{E}$ is a list $((x_i)_{i\in I}, (y_j)_{j\in J})$ of vectors such that $x_i \in X_i$ for each i and $y_j \in Y_j$ for each j. The allocation is said to be *attainable* (or *admissible* or even *feasible*) if supply equals demand. That is,

$$\sum_{i\in I} x_i = \omega + \sum_{j\in J} y_j\,.$$

The collection of all attainable allocations is denoted $\mathcal{A}$, i.e.,

$$\mathcal{A} = \Big\{((x_i),(y_j)) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j\colon\ \sum_{i\in I} x_i = \omega + \sum_{j\in J} y_j\Big\}\,.$$

An economy is said to be *convex* if all consumption sets, all preferences, and all productions sets are convex. An economy is said to be *finite* if the sets I, J of consumers and producers are finite and if there are finitely many commodities ℓ, or equivalently if the commodity space is finite dimensional, say $L = \mathbb{R}^\ell$ ordered by the standard ordering.

A *private ownership economy* is an economy $\mathcal{E}$ in which it is further assumed that:

- Each consumer $i \in I$ has an initial endowment ω_i and $\sum_{i\in I} \omega_i = \omega$.

- Each consumer $i \in I$ has a *share* $\theta_{ij} \geq 0$ in the profit of the producer j, where for each j we have $\sum_{i\in I} \theta_{ij} = 1$.

2.5. Equilibrium and optimality

We start with the fundamental economic notion of price decentralization which will be formulated as a geometric notion of supporting an allocation by prices.

DEFINITION 1. *An attainable allocation* $((x_i),(y_j))$ *is said to be:*

1. *a* ***weak valuation equilibrium****, if there exists a (non-zero) price* p *that* supports *the allocation in the sense that it satisfies the following properties.*

   a) For each consumer i*:* $x \succ_i x_i$ *implies* $p \cdot x \geq p \cdot x_i$.

   b) For each producer j*:* $y \in Y_j$ *implies* $p \cdot y \leq p \cdot y_j$.

2. *a **valuation equilibrium**, if there exists a (non-zero) price p that in addition to satisfying properties (a) and (b) above, it also satisfies the property:*

   c) $p \cdot \omega \neq 0$.

REMARK 1. Property (c) above is important for the infinite dimensional case. If the commodity space does have strictly positive vectors (for instance let $L = ca[0,1]$, the Riesz space of all regular Borel measures on $[0,1]$), then every attainable allocation is a weak valuation equilibrium.

The preceding definition will be generalized later in two ways. First we shall allow for prices valuation function that need not be linear. That is, we assumed above that the valuation function $p: L \to \mathbb{R}$ is linear and we shall later consider a case where the valuation function, denoted $\mathcal{R}_p: L \to \mathbb{R}$, may not be linear. The above definitions of supporting prices and valuation equilibria, remain the same, replacing p by $\mathcal{R}_p$. Second, in the absence of convexity assumptions on the production sets Y_j and the preferred sets, we shall weaken the conditions on the supporting price by only assuming that the "necessary condition" for profit maximization and expenditure minimization hold (in a precise mathematical sense to be defined later).

Let us move to notions of economic optimality whose definitions do not depend on market prices. An attainable allocation $((x_i),(y_j))$ is:

- **Pareto optimal**, if there is no attainable allocation $((x'_i),(y'_j))$ satisfying $x'_i \succeq_i x_i$ for all i and $x'_i \succ_i x_i$ for some i.

- **weakly Pareto optimal**, if there does not exist an attainable allocation $((x'_i),(y'_j))$ with $x'_i \succ_i x_i$ for all i.

Two of the classical propositions in economics relate the notions optimality and supporting prices. They are known as the first and second welfare theorems. These theorems were first proven in reasonable generality by Arrow [16] and Debreu [51]. Subject to appropriate conditions, the two theorems can be stated as follows:

1$^{\text{st}}$ Welfare Theorem: *Every valuation equilibrium is Pareto optimal.*

2$^{\text{nd}}$ Welfare Theorem: *Every Pareto optimal allocation is a valuation equilibrium.*

The importance of the welfare theorems is that they characterize the supportability by market prices in terms of normative notions.

THEOREM 1 (First Welfare Theorem). *If in a production economy,*

(a) each consumption set is convex and contains zero,

(b) each production set contains zero, and

(c) preferences are lower semicontinuous,

then every valuation equilibrium having a supporting price satisfying $p \cdot \omega > 0$ [6] *is weakly Pareto optimal.*

Proof. Let $((x_i), (y_j))$ be a valuation equilibrium. This means that it is an attainable allocation supported by a non-zero price p that satisfies $p \cdot \omega > 0$. Suppose by way of contradiction that $((x_i), (y_j))$ is not weakly Pareto optimal. Then, there exists some attainable allocation $((x'_i), (y'_j))$ such that $x'_i \succ x_i$ for each i.

The supporting property implies that $p \cdot x'_i \geq p \cdot x_i$ for each i and $-p \cdot y'_j \geq -p \cdot y_j$ for each j. Therefore,

$$p \cdot \omega = \sum_{i \in I} p \cdot x'_i - \sum_{j \in J}^{k} p \cdot y'_j \geq \sum_{i \in I}^{m} p \cdot x_i - \sum_{j \in J}^{k} p \cdot y_j = p \cdot \omega \,.$$

Hence, $p \cdot x'_i = p \cdot x_i$ for each i and $p \cdot y'_j = p \cdot y_j$ for each j. From the condition $0 \in Y_j$, we see that $p \cdot y_j \geq 0$ for each j. Consequently, we have $\sum_{i \in I} p \cdot x_i = \sum_{j \in J} p \cdot y_j + p \cdot \omega > 0$, hence there exists some i such that $p \cdot x'_i = p \cdot x_i > 0$.

Using that the preference $\succeq_i$ is lower semicontinuous and that zero belongs to X_i, we conclude that there exists some $\delta \in (0,1)$ such that $\delta x'_i \succ_i x_i$. This implies

$$\delta p \cdot x_i = \delta p \cdot x'_i = p \cdot (\delta x'_i) \geq p \cdot x_i > 0 \,,$$

which is not possible. ■

[6] We first notice that $p \cdot \omega \geq 0$ (and $p \geq 0$) when $\omega \in L_+$ and the preferences are strictly monotonic. Thus the assumption $p \cdot \omega > 0$ will be met under one of the additional assumption that will be considered later in this paper (i) $\omega \in \text{int}\, L_+$ and $p \neq 0$, and (ii) $p \cdot \omega \neq 0$.

3. Convex exchange economies

3.1. The positive cone L_+ has a nonempty interior

We begin by presenting a version of the second welfare theorem when the positive cone L_+ has a nonempty interior, a result due to Debreu [51] for infinite dimensional commodity spaces.

THEOREM 2 (Second Welfare Theorem). *If in an exchange economy the positive cone L_+ has a nonempty interior, every strict preference is monotone and convex, then every weakly Pareto optimal (and hence every Pareto optimal) allocation $((x_i),(y_j))$ is a weak valuation equilibrium.*

Moreover, if the total endowment ω is an interior point of L_+ and p is the supporting price, we can assume that $p\cdot\omega>0$ (in which case every weakly Pareto optimal allocation is a valuation equilibrium).

Proof. Let $((x_i),(y_j))$ be a weakly Pareto optimal allocation, we first claim that

$$\omega \notin \Big[\sum_{i\in I} P_i(x_i)\Big] + \operatorname{Int} L_+ \ \text{ where } \ P_i(x_i)=\{y\in L_+ : y\succ_i x_i\}.$$

Indeed, if it were not true, then we could write $\omega=\sum_{i\in I}[x_i'+u]$, for some $x_i'\in P_i(x_i)$ $(i\in I)$ and some $u\in\operatorname{Int} L_+$. But then for each consumer i the strict monotonicity of P_i implies $x_i'+u\in P_i(x_i)$, contradicting the weak Pareto optimality of the allocation $((x_i),(y_j))$.

Clearly the set $\sum_{i\in I} P_i(x_i)+\operatorname{Int} L_+$ is convex (since each $P_i(x_i)$ is convex by assumption and $\operatorname{Int} L_+$ is convex, hence the sum is also convex), open, and nonempty (since each $P_i(x_i)$ is nonempty from the monotonicity assumption, and $\operatorname{Int} L_+$ is also nonempty). Consequently, by the Hahn–Banach separation theorem, there exists a non-zero functional $p\in L'$ such that for any choices of $y_i\in P_i(x_i)$ for each $i\in I$ and all $u\in\operatorname{Int} L_+$ we have

$$\sum_{i\in I} p\cdot y_i + p\cdot u \ge p\cdot\omega = p\cdot\Big(\sum_{i\in I} x_i\Big). \qquad (\star)$$

From the continuity of the functional $p\in L'$, we deduce that $(\star)$ is still valid for every $y_i\in\operatorname{cl} P_i(x_i)$ for each $i\in I$ and every $u\in L_+$. Noticing that the monotonicity assumption implies $x_i\in\operatorname{cl} P_i(x_i)$ for each $i\in I$ and $0\in L_+$, we infer that $p\cdot y_i\ge p\cdot x_i$ for every $y_i\in P_i(x_i)$ for each $i\in I$, and $p\cdot u\ge 0$ for every $u\in L_+$. Hence $p>0$ and consequently $((x_i),(y_j))$ is a valuation equilibrium.

If, in addition we assume that $\omega \in \operatorname{Int} L_+$, then we can immediately deduce that $p \cdot \omega > 0$. ■

When ω is not an interior point of L_+ the preceding theorem is false even in the finite dimensional case with one consumer.

EXAMPLE 1 (Arrow [16]). Consider an exchange convex economy with one consumer and two commodities. The consumer has the utility function $u(x, y) = \sqrt{x} + \sqrt{y}$ and the initial endowment $\omega = (0, 1)$. Clearly, ω is the only Pareto optimal allocation. A straightforward verification shows that ω is a weak valuation equilibrium but not a valuation equilibrium, that is, there is no supporting price p such that $p \cdot \omega > 0$. ■

This problem can be solved by requiring that ω is an interior point of the positive orthant of $\mathbb{R}^2$. However, the standard infinite dimensional setting is one where this problem cannot be readily assumed away. When ω is not an interior point we need to add extra conditions on preferences and use the lattice structures of the commodity and price spaces. The main definitions and properties that are needed hereafter are recalled in the next section.

3.2. Commodity and price spaces as vector lattices

This section considers the case where the positive cone L_+ may have an empty interior. We shall need to introduce first a lattice structure on both the commodity and the price spaces and second a condition on the consumers known as "properness."

A *Riesz space* (or a *vector lattice*) is an ordered vector space L with the additional property that the supremum (least upper bound) of every nonempty finite subset of L exists. Following the classical lattice notation, we denote the supremum of the set $\{x, y\}$ by $x \vee y$. If now $\{x_i : i \in I\}$ is a finite collection of elements in a Riesz space, then we shall denote their supremum by $\bigvee_{i \in I} x_i$, that is,

$$\bigvee_{i \in I} x_i = \sup \{x_i\colon\ i \in I\}\,.$$

In modern equilibrium theory quite often the commodity-price duality is represented by a dual pair $\langle L, L' \rangle$, where L is a Riesz space and L' is also a Riesz space. For this we need to introduce some more terminology. In an ordered vector space L an *interval* is any subset of L of the form $[x, y] = \{z \in L\colon\ x \le z \le y\}$. A set is said to be *order bounded* if it is included in an order interval. A linear functional

$p: L \to \mathbb{R}$ on an ordered vector space is said to *order bounded* if it carries order bounded sets to bounded subsets of $\mathbb{R}$.

We define the *order dual* of L, denoted by $L^\sim$, as the collection of all order bounded linear functionals on L. The order dual $L^\sim$ is ordered by the cone of positive linear functionals and is itself an ordered vector space, provided that the cone L_+ is generating.[7] Recall that a linear functional $p: L \to \mathbb{R}$ is said to be *positive* if $x \geq 0$ implies $p \cdot x \geq 0$.

The following theorem which is the basis of our analysis (see [6] and [7]).

THEOREM 3 (Riesz–Kantorovich). *If L is a Riesz space, then $L^\sim$ is also a Riesz space. In particular, if $\{p_i\colon\ i \in I\}$ is a finite collection in $L^\sim$, then for each $x \in L_+$ we have the following famous Riesz–Kantorovich formula:*

$$\Big[\bigvee_{i\in I} p_i\Big](x) = \sup\Big\{\sum_{i\in I} p_i \cdot x_i\colon\ x_i \in L_+ \text{ and } \sum_{i\in I} x_i = x\Big\}.$$

We need now to apply the above Riesz–Kantorovich formulas to prices in the topological dual L'. For this we shall assume that (L, τ) is an ordered topological space and that L' is a vector sublattice (Riesz subspace) of $L^\sim$, that is if p, q are in L', then the supremum $p \vee q$ (which exist in $L^\sim$) also belongs to L'. We point out that if L *locally solid*,[8] then L' is automatically a vector sublattice of $L^\sim$. For details of the lattice structure of the topological and order duals of a topological Riesz space see again [6] and [7].

We now present the "properness" condition (see Mas-Colell [81] for the standard reference) and we shall use here the one taken from Tourky [109].

DEFINITION 2 (Tourky). *A preference correspondence $P: L_+ \twoheadrightarrow L_+$ is said to be ω-**proper** if there exists some correspondence $\widehat{P}: L_+ \twoheadrightarrow L$ (which is assumed to be convex-valued if P is convex-valued) such that for each $x \in L_+$:*

1. *the vector $x + \omega$ is an interior point of $\widehat{P}(x)$; and*

[7] If L is an ordered vector space with cone L_+, then L_+ is said to be **generating** if the vector space generated by L_+ is all of L, or equivalently if $L = L_+ - L_+$. If L is also equipped with a Hausdorff locally convex topology, then the dual "cone" $L'_+ = \{p \in L'\colon\ p \cdot x \geq 0\ \ \forall\, x \in L_+\}$ is indeed a cone that is also weakly closed cone. The cone of any Riesz space is generating.

[8] A subset A in a Riesz space is said to be *solid* if $|x| \leq |y|$ and $y \in A$ imply $x \in A$. A linear topology is called *locally solid* if it has a base at zero consisting of solid sets.

2. $\widehat{P}(x) \cap L_+ = P(x)$.

We are now ready to state a version of the second welfare theorem when both the commodity space and the price space are vector lattices.

THEOREM 4 (Mas-Colell–Richard [82]). *Assume that L is a vector lattice and that L' is a vector sublattice of the order dual $L^\sim$ of L. If in an exchange economy preferences are ω-proper, have convex strict upper sections, and are strictly monotone, then every weakly Pareto optimal allocation is a valuation equilibrium.*

Proof. Let $(x_i)_{i\in I}$ be a weakly Pareto optimal allocation. For each i, let $\widehat{P}_i(x_i)$ be the convex extension of the set $P_i(x_i) = \{y \in L_+\colon\ y \succ_i x_i\}$ given by the ω-properness assumption. Let $\mathcal{A}$ be the convex non-empty set of all attainable allocations, i.e.,

$$\mathcal{A} = \Big\{(y_i) \in L_+^I\colon\ \sum_{i\in I} y_i = \omega\Big\}.$$

Since $(x_i)_{i\in I}$ is weakly Pareto optimal, it follows that

$$\mathcal{A} \bigcap \prod_{i\in I} \widehat{P}_i(x_i) = \emptyset\ .$$

Since, according to the ω-properness assumption, $(x_i + \omega)_{i\in I}$ is an interior point of $\prod_{i\in I} \widehat{P}_i(x_i)$, it follows that we can separate the two convex sets $\mathcal{A}$ and $\prod_{i\in I} \widehat{P}_i(x_i)$ of the locally convex space L. That is, by the separation theorem, there exists a non-zero functional $(p_i)_{i\in I} \in (L')^I$ such that for every $(z_i)_{i\in I} \in \prod_{i\in I} \widehat{P}_i(x_i)$ we have

$$\sup\Big\{\sum_{i\in I} p_i \cdot y_i\colon\ (y_i) \in \mathcal{A}\Big\} \le \sum_{i\in I} p_i \cdot z_i\,. \qquad (\dagger)$$

The preceding inequality in conjunction with the continuity of the functionals $p_i \in L'$ shows that ($\dagger$) is also valid for every $(z_i)_{i\in I}$ in $\prod_{i\in I} \operatorname{cl} \widehat{P}_i(x_i)$. Also, from the strict monotonicity of the preferences we obtain $x_i \in \operatorname{cl} \widehat{P}_i(x_i)$ for each consumer i. Consequently,

$$\sup\Big\{\sum_{i\in I} p_i \cdot y_i\colon\ (y_i) \in \mathcal{A}\Big\} = \sum_{i\in I} p_i \cdot x_i, \qquad (\dagger\dagger)$$

and

$$y \succ_i x_i \quad \Longrightarrow \quad p_i \cdot y \ge p_i \cdot x_i\,. \qquad (\dagger\,\dagger\,\dagger)$$

Next, we define the function $p\colon\ L \to \mathbb{R}$ by

$$p(y) = \sup\Big\{\sum_{i\in I} p_i \cdot y_i \colon\ y_i \in L_+ \text{ and } \sum_{i\in I} y_i = y\Big\}\,.$$

Clearly, from the lattice assumptions made on L and L', we deduce that $p = \bigvee_{i\in I} p_i \in L'$; see Theorem 3. Therefore,

$$\sum_{i\in I} p_i \cdot x_i = p(\omega) = \sum_{i\in I} p(x_i)\,.$$

Furthermore, for every $i \in I$ we have $p(x_i) \geq p_i \cdot x_i$, and consequently $p(x_i) = p_i \cdot x_i$ for each $i \in I$. Therefore, from $(\dagger\dagger\dagger)$ we get:

$$y \succ_i x_i \quad\Longrightarrow\quad p(y) \geq p_i \cdot y \geq p_i \cdot x_i = p(x_i)\,,$$

which shows that the price p supports the allocation $(x_i)_{i\in I}$.

We end the proof by showing that the price p is non-zero, and in fact that $p(\omega) > 0$. Indeed, we know that $(x_i + \omega)_{i\in I}$ is an interior point of $\prod_{i\in I} \widehat{P}_i(x_i)$. So by $(\dagger)$ and $(\dagger\dagger)$, $\sum_{i\in I} p_i \cdot x_i < \sum_{i\in I} p_i \cdot (x_i + \omega_i)$. Hence, $\sum_{i\in I} p_i \cdot \omega > 0$. Again we have $p(\omega) \geq p_i \cdot \omega$ for every $i \in I$. Consequently, $p(\omega) \geq \frac{1}{\#I} \sum_{i\in I} p_i \cdot \omega > 0$. ■

3.3. Finite dimensional commodity spaces ordered by non-lattice cones

A recurrent theme in the literature concerns the sharp contrast between the central role of vector lattices in infinite dimensional theory and the apparent irrelevance of lattice theoretic properties in finite dimensional analysis. This contrast was highlighted by Mas-Colell [81, p. 506] in the following quote:

> "A major surprise of this paper is precisely this relevance of lattice theoretic properties to the existence of equilibrium problem. One would not have been led to expect it from the finite dimensional theory. In the latter it is possible to formalize and solve the existence problem using only the topological and convexity structures of the space (cf. Debreu [54]). Informally, a source for the difference seems to be the following: in general vector lattices, order intervals (i.e., sets of the form $\{x \colon a \leq x \leq b\}$) are, as convex sets, much more tractable and well behaved than general bounded, convex sets."

Mas-Colell's quote has posed a long standing open question concerning the application of the "lattice" approach to finite dimensional models:

- *Does Theorem* 4 *hold true when the commodity space is finite dimensional and ordered by a non-lattice cone?*

The next example, due to Monteiro and Tourky [89], shows that the Mas-Colell–Richard theorem does not hold for finite dimensional commodity spaces when ordered by a non-lattice cone.

EXAMPLE 2 (Monteiro–Tourky). This is an example of an exchange economy with two consumers and three commodities, i.e., $L = \mathbb{R}^3$. The commodity space L is ordered with the order induced by the "ice cream" cone:

$$\begin{aligned} L_+ &= \Big\{(x,y,z) \in \mathbb{R}^3\colon\ z \geq \sqrt{x^2+y^2}\,\Big\} \\ &= \{\lambda(x,y,1)\colon\ \lambda \geq 0 \ \text{ and } \ x^2+y^2 \leq 1\}. \end{aligned}$$

Of course, the "ice cream" cone does not induce a lattice structure on L. But, nevertheless, L_+ is a closed and generating cone—and so the order intervals are compact sets.

The initial endowment of consumer 1 is $\omega_1 = (0,\alpha,\alpha) \in L_+$, where $0 < \alpha < 1$ is a fixed number. The initial endowment of consumer 2 is $\omega_2 = (0, 1-\alpha, 1-\alpha) \in L_+$. Therefore, the total endowment of the exchange economy is $\omega = (0,1,1)$. Since ω defines an extremal ray, it follows that $[0,\omega] = \{\beta\omega\colon\ 0 \leq \beta \leq 1\} = \{(0,\beta,\beta)\colon\ 0 \leq \beta \leq 1\}$.

The utility functions of the consumers are defined on L by

$$u_1(x,y,z) = 2z \quad \text{ and } \quad u_2(x,y,z) = x + 2z\,.$$

It is easy to check that these utility functions are linear, strictly monotone, and obviously ω-uniformly proper.

Let us say that an allocation $x = (x_1, x_2)$ is *non-trivial* if $x_i \neq 0$ for $i = 1,2$. We conclude the example by establishing the following surprising result.

- *No non-trivial attainable allocation is a valuation equilibrium. In particular, this economy does not have any competitive eqilibria.*[9]

To establish this claim, let $x = (x_1, x_2)$ be an arbitrary non-trivial attainable allocation. Therefore, there exists some $0 < \alpha < 1$ such that $x_1 = (0,\alpha,\alpha)$ and $x_2 = (0, 1-\alpha, 1-\alpha)$. Now assume by way of contradiction that there exists some $p \in L'$ with $p\cdot\omega \neq 0$ that supports the above allocation. From the strict monotonicity of preferences, it must be the case that $p \in L'_+$. We can therefore assume without loss of generality that $p\cdot\omega = 2$. This implies that $p\cdot x_1 = 2\alpha$ and $p\cdot x_2 = 2-2\alpha$.

We claim that $p \geq (1,0,2)$ and that $p \geq (0,0,2)$ in the order induced by L_+. That is, for every bundle $a = (x,y,z) \in L_+$ we have

$$p \cdot a \geq \max\{x + 2z, 2z\}\,.$$

[9] Recall that an allocation is a *competitive* (or a *Walrasian*) *equilibrium* if $x \succ_i x_i$ for any consumer i implies $p \cdot x > p \cdot \omega_i$.

To see this, let $a = (x, y, z)$ be an arbitrary point in L_+ and notice that $\max\{x + 2z, 2z\} \geq 0$. If $\max\{x + 2z, 2z\} = 0$ is the case, then it follows that $x = y = z = 0$, and so $p \cdot a = 0 = \max\{x + 2z, 2z\}$, as required. Now consider the case $\max\{x + 2z, 2z\} > 0$. This implies $z > 0$. From $u_1(\frac{\alpha}{z}a) = 2\alpha = u_1(x_1)$, the strict monoticity of u_1 and the supporting property of p, we get $p \cdot (\frac{\alpha}{z}a) \geq p \cdot x_1 = 2\alpha$ and so $p \cdot a \geq 2z$. If $x + 2z \leq 0$, then $p \cdot a \geq 2z = \max\{x + 2z, 2z\}$. Finally, we consider the case $x + 2z > 0$. In this case, we have $u_2(\frac{2-2\alpha}{x+2z}a) = 2 - 2\alpha = u_2(x_2)$. This implies $p \cdot \left(\frac{2-2\alpha}{x+2z}a\right) \geq p \cdot x_2 = 2 - 2\alpha$ and so $p \cdot a \geq x + 2z$, as required.

Now let $q = p - (0, 0, 2) \in L_+$ and note that the vector

$$(x_0, y_0, z_0) = q - (1, 0, 0) = p - (1, 0, 2) \in L_+$$

satisfies the property

$$y_0 + z_0 = (x_0, y_0, z_0) \cdot \omega = [p - (1, 0, 2)] \cdot \omega = p \cdot \omega - 2\alpha = 0\,.$$

This implies that $(1+x_0, -z_0, z_0) = (x_0, y_0, z_0) + (1, 0, 0) = q \in L_+$, and thus $1 + x_0 = 0$. Consequently, $x_0 = -1$ and from this it follows that $(x_0, y_0, z_0) = (-1, -z_0, z_0) \notin L_+$, a contradiction. This contradiction shows that the allocation (x_1, x_2) cannot be supported by any price. ■

3.4. Optimality with nonlinear prices

With the preceding example in mind, Aliprantis, Tourky, and Yannelis [13] extended the theory of general equilibrium beyond vector lattices by introducing the notion of a personalized price system that induces non-linear value functions. We briefly illustrate their optimality results below.

We call an arbitrary linear functional $p = (p_1, p_2, \ldots, p_m)$ on L^m a *list of personalized prices* (or simply a *list of prices*). For each commodity bundle $x \in L_+$, we let $\mathcal{A}_x$ denote the set of all allocations when the total endowment is x, i.e.,

$$\mathcal{A}_x = \Big\{ y = (y_1, y_2, \ldots, y_m) \in \prod_{i=1}^{m} X_i\colon\ \sum_{i=1}^{m} y_i \leq x \Big\}\,.$$

DEFINITION 3. *The* ***generalized price*** (*or the* ***Riesz–Kantorovich formula***) *of an arbitrary list of personalized prices* $p = (p_1, p_2, \ldots, p_m)$ *is the function* $\mathcal{R}_p\colon L_+ \to [0, \infty]$ *defined by*

$$\mathcal{R}_p(x) = \sup_{y \in \mathcal{A}_x} \left[\, p_1 \cdot y_1 + p_2 \cdot y_2 + \cdots + p_m \cdot y_m \,\right].$$

Clearly, if $p \in (L^\sim)^m$, then $\mathcal{R}_p$ is a real-valued function. The value $\mathcal{R}_p(x)$ is the maximum value that one can obtain by decomposing the bundle x into consumable allocations, where each consumer i pays the price $p_i \cdot x_i$ for her assigned bundle x_i.

The basic properties of the generalized prices are included in the next result that can be found in [13].

LEMMA 1. *If $p = (p_1, p_2, \ldots, p_m) \in (L^\sim)^m$ is a list of order bounded personalized prices, then its generalized price $\mathcal{R}_p \colon L_+ \to [0, \infty)$ is a non-negative real-valued function such that:*

1. *$\mathcal{R}_p$ is monotone, i.e., $x, y \in L_+$ with $x \le y$ implies $\mathcal{R}_p(x) \le \mathcal{R}_p(y)$.*
2. *$\mathcal{R}_p$ is super-additive, that is, $\mathcal{R}_p(x) + \mathcal{R}_p(y) \le \mathcal{R}_p(x + y)$ for all $x, y \in L_+$.*
3. *$\mathcal{R}_p$ is positively homogeneous, that is, $\mathcal{R}_p(\alpha x) = \alpha(\mathcal{R}_p(x))$ for all $\alpha \ge 0$ and $x \in L_+$.*
4. *If $p_1 = p_2 = \cdots = p_m = q \ge 0$, then $\mathcal{R}_p(x) = q \cdot x$ for all $x \in L_+$.*
5. *If $x \in L_+$, then $p_i \cdot x \le \mathcal{R}_p(x)$.*
6. *If L is a vector lattice (or more generally, if L has the Riesz Decomposition property[10] and its cone is generating), then for each $x \in L_+$ we have $\mathcal{R}_p(x) = \pi(x)$, where $\pi = \bigvee_{i=1}^m p_i$.*

The next definition generalizes the notion of a supportability.

DEFINITION 4. *An allocation $(x_1, x_2, \ldots, x_m)$ is said to be a* ***personalized valuation equilibrium*** *if there exists some list of personalized prices $p = (p_1, p_2, \ldots, p_m) \in (L^\sim)^m$ such that:*

1. *$\mathcal{R}_p(\omega) > 0$,*
2. *$y \in P_i(x_i) \implies \mathcal{R}_p(y) \ge \mathcal{R}_p(x_i)$, and*
3. *the following arbitrage–free condition holds*

$$\mathcal{R}_p(\omega) = \sum_{i=1}^m \mathcal{R}_p(x_i)\,.$$

It turns out that the properties of generalized prices that are listed in Lemma 1 can be easily used to extend the proofs of Theorems 8 and Theorem 9 to the case of ordered vector spaces with non-linear prices. We conclude this section by stating an extension of the first and second welfare theorems to ordered vector spaces.

[10] That is, $[0, x] + [0, y] = [0, x + y]$ for all $x, y \in L_+$

THEOREM 5. *In an exchange economy every valuation equilibrium is a personalized valuation equilibrium. Furthermore, for an ordered topological vector space L with topologically bounded order intervals we have the following.*

1. *If preferences are ω-proper, have convex strict upper sections, and are strictly monotone, then every weakly Pareto optimal allocation is a personalized valuation equilibrium.*

2. *If preferences have open upper sections, then every personalized valuation equilibrium is weakly Pareto optimal.*

3. *If L is a vector lattice and L' is a vector sublattice of the order dual $L^{\sim}$ of L, then a personalized valuation equilibrium is a valuation equilibrium.*

For a proof of this result and more on personalized prices, we refer the reader to [13].

4. Non-convex production economies

In this section, we no longer assume that the production sets Y_j are convex and the profit maximization behavior of the firms is replaced by the so-called *marginal pricing rule*. We consider a finite dimensional setting, that is, the economy has finitely many commodities ℓ, or, equivalently, the commodity space $L = \mathbb{R}^\ell$.

The firm j is said to follow the *marginal pricing rule* if at equilibrium the production y_j satisfies the *first order necessary condition* for profit maximization (for the fixed price p) on the production set Y_j, in a precise mathematical sense formalized below. As we shall see, if the production set Y_j is additionally assumed to be convex, then the above necessary condition will also be sufficient. In other words, in the convex case, the marginal pricing rule coincides with profit maximization.

4.1. Marginal Pricing and Normal Cones

We now give a precise mathematical definition of the *marginal pricing rule*, with the help of the following normal cones. Let C be a nonempty subset of $\mathbb{R}^\ell$, we call **perpendicular vector** (or **proximal normal vector**) to C at $x \in \operatorname{cl} C$, every vector p in the set

$$\perp_C(x) := \left\{ p \in \mathbb{R}^\ell \colon\ \exists\, \rho > 0,\ \forall\, x \in \operatorname{cl} C,\ p \cdot x \geq p \cdot x' - \rho \|x' - x\|^2 \right\},$$

and **limiting normal vector to C at x**, every vector p in the set

$$\widehat{N}_C(x) := \left\{ \lim_{n\to\infty} p^\nu \colon\ \exists\,\{x^\nu\} \subseteq \mathrm{cl}\,C,\ x^\nu \to x \text{ and } \forall\ \nu,\ p^\nu \in \perp_C (x^\nu) \right\}.$$

It is worth pointing out that we have not assumed above that the set C is closed. The closedness of C is quite common when $C = Y_j$ is a production set, but it is no longer the case when we consider the preferred set $C = P_i(x_i)$ (see below). From the above definitions, one easily sees that:

$$\perp_{\mathrm{cl}\,C}(x) = \perp_C(x) \subseteq \widehat{N}_C(x) = \widehat{N}_{\mathrm{cl}\,C}(x) \quad \text{for every} \quad x \in \mathrm{cl}\,C.$$

DEFINITION 5. *The producer j, with production set $Y_j \subseteq \mathbb{R}^\ell$, is said to follow the* marginal pricing rule *if its equilibrium condition is formalized by the condition*

$$p \in \widehat{N}_{Y_j}(y_j).$$

The term "marginal pricing rule" is justified by the fact that the stronger condition $p \in \perp_{Y_j} (y_j)$ [and thus also the weaker condition $p \in \widehat{N}_{Y_j}(y_j)$] is a first order necessary condition for profit maximization. Indeed, if, for a given price p, the production y_j maximizes the profit $p \cdot y_j'$ over the production set Y_j, then, for every $\rho \geq 0$, it also maximizes the function $p \cdot y_j' - \rho\|y_j' - y_j\|^2$ over the set Y_j; hence, the condition $p \in \perp_{Y_j} (y_j)$ is satisfied. We shall see below that, if Y_j is convex, we can take $\rho = 0$ in the cone $\perp_{Y_j} (y_j)$, that is, the necessary condition $p \in \perp_{Y_j}(y_j)$ for profit maximization is also sufficient.

REMARK 2. The condition $p \in \perp_{Y_j} (y_j)$ has a clear economic and geometric interpretation. It means that, given the price system p, the plan y_j maximizes the quadratic function $p \cdot y_j - \rho\|y_j' - y_j\|^2$ over the production set Y_j, i.e., it maximizes the profit $p \cdot y_j$ up to the "perturbation" $-\rho\|y_j' - y_j\|^2$. It can also be interpreted in terms of "non-linear prices" by noting that it is equivalent to saying that y_j maximizes the quadratic function $\pi_j(y_j') = [p - \rho(y_j' - y_j)] \cdot (y_j' - y_j)$ over the production set Y_j.

Moreover, the condition $p \in \perp_{Y_j}(y_j)$ formalizes in a natural way the notion of "orthogonality" to a set. Indeed, it is easy to show that it is equivalent to the following geometric condition:

$$(\perp') \qquad \exists\ \varepsilon > 0 \quad \text{such that} \quad \mathrm{int}[B(y_j + \varepsilon p, \varepsilon\|p\|)] \cap \mathrm{cl}\,Y_j = \emptyset\,.$$

The perpendicularity condition, however, is too restrictive in many situations, i.e., it is easy to find examples for which $y_1 \in \partial Y_1$ and $\perp_{Y_1}(y_1) = \{0\}$. This is essentially the reason why, in the following, we need to formalize the marginal pricing rule with the the weaker notion of limiting normal cone.

4.2. Properties of the Normal Cones

The following propositions summarize the main properties of the normal cones. A general reference on this subject is Clarke [43].

PROPOSITION 6. *Let C be a nonempty closed subset of $\mathbb{R}^\ell$.*

1. *For every $x \in C$, the sets $\perp_C(x)$, and $\widehat{N}_C(x)$ are closed cones with vertex 0, and $\perp_C(x) \subseteq \widehat{N}_C(x)$.*
2. *Let U be an open subset of $\mathbb{R}^\ell$ and let $x \in C \cap U$, then one has $\perp_{C\cap U}(x) = \perp_C(x)$, and $\widehat{N}_{C\cap U}(x) = \widehat{N}_C(x)$.*
3. *If $x \in \partial C$, then $\widehat{N}_C(x) \neq \{0\}$, but one may have $\perp_C(x) = \{0\}$. Conversely, if $\widehat{N}_C(x) \neq \{0\}$, then $x \in \partial C$.*
4. *Let C_k be a finite family of closed subsets of $\mathbb{R}^\ell$ $(k \in K)$, let U be an open set containing $C := \Pi_{k\in K} C_k$ and let $f: U \to \mathbb{R}$ be a differentiable function at a point $x = (x_k)_{k\in K}$ in C (resp. twice continuously differentiable on a neighborhood of x). If x is a minimum of the function f over the set C, then the following necessary condition is satisfied:*

$$-\nabla_{x_k} f(x) \in \widehat{N}_{C_k}(x_k)\ [\text{resp. } -\nabla_{x_k} f(x) \in \perp_{C_k}(x_k)\ ]\ \text{ for all }\ k \in K.$$

REMARK 3. It is essentially because of the above Property 2, that the cones $\widehat{N}_{Y_j}(\cdot)$ is well adapted to price decentralization by the marginal pricing rule. In other words, if Y_j is closed, to every element $y_j \in \partial Y_j$ [which coincides with the set of weakly efficient production plans, under Assumption (P)], one can associate a non-zero price p that supports the plan y_j, in the sense that $p \in \widehat{N}_C(x)$. However, this property does not hold, in general, for the cone of perpendicular vectors.

The next proposition states that the two definitions of normal cones coincide with the classical notion when C is convex or when C has a smooth boundary.

PROPOSITION 7. *For a subset C of $\mathbb{R}^\ell$ the following holds.*

1. *If C is closed and convex, then, for each $x \in C$, we have*
$$\perp_C(x) = \widehat{N}_C(x) = \{p \in \mathbb{R}^\ell\colon\ p \cdot x \geq p \cdot x'\ \forall\, x' \in C\}\,.$$
2. *If $C = \{x' \in \mathbb{R}^\ell\colon\ g(x') \leq 0\}$, where the function $g: \mathbb{R}^\ell \to \mathbb{R}$ is twice continuously differentiable and satisfies the nondegeneracy assumption $[g(x) = 0 \implies \nabla g(x) \neq 0]$, then*
$$\perp_C(x) = \widehat{N}_C(x) = \{\lambda \nabla g(x)\colon\ \lambda \geq 0\}\,.$$

3. If C satisfies Free Disposal, that is, $C - \mathbb{R}^\ell_+ \subseteq C$, then, for every $x \in C$ we have $\widehat{N}_C(x) \subseteq \mathbb{R}^\ell_+$.

4.3. The case of convex strict preferences

Consider, as in Section 2.4, the production economy

$$\mathcal{E} = (\mathbb{R}^\ell, (X_i, \omega_i \succeq_i)_{i \in I}, (Y_j)_{j \in J}),$$

and define for each consumer i and each $x = (x_i) \in \prod_{i \in I} X_i$, the strictly preferred sets

$$P_i(x) = \{x'_i \in X_i \colon\ x'_i \succ x_i\}.$$

We now consider the case of convex strict preferences for which we can state the following results.

THEOREM 8. *Let $((x_i), (y_j))$ be a weakly Pareto optimal allocation such that, for each consumer i, the set $P_i(x_i)$ is convex, $x_i \in \operatorname{cl} P_i(x_i)$ [which holds under the Nonsatiation Assumption], and, for every producer j, Y_j is closed. Then there exists a non-zero price p such that:*

- *For each consumer i, $x \succ x_i$ implies $p \cdot x \geq p \cdot x_i$.*
- *For each producer j, $p \in \widehat{N}_{Y_j}(y_j)$.*

The proof of Theorem 8 is given in the next section. The interpretation of this result is the same as for the second welfare theorem of convex economies, the only difference being that here the producers strive to follow the marginal pricing rule rather than maximizing their profits as in the convex case.

REMARK 4. The statement of Theorem 8 does not treat symmetrically the consumption and the production sectors. Indeed, the presence of non-convexities in these two sectors is of a different nature. For example, by considering a continuum of consumers all infinitely small (i.e. a measure space without atoms), the convexity assumption on consumers' preferences can be dropped (see Aumann [19], Hildenbrand [66]). An analogous treatment of the production sector, however, is not very realistic on economic grounds, since the presence of increasing returns (and other types of non-convexities) usually characterizes large firms. ∎

We now state the standard version of the second theorem of welfare economics in the convex case (compare with the statement given in Section 3).

COROLLARY 9. *Let $((x_i),(y_j))$ be a weakly Pareto optimal allocation such that for each consumer i, the set $P_i(x_i)$ is convex, $x_i \in \operatorname{cl} P_i(x_i)$ [which holds under the Nonsatiation Assumption], and, for each producer j, Y_j is closed and* **convex**. *Then, there exists a non-zero price p such that:*

- *For each consumer i, $x \succ x_i$ implies $p \cdot x \geq p \cdot x_i$.*
- *For each producer j, and each $y_j \in Y_j$ we have $p \cdot y_j \geq p \cdot y_j$.*

REMARK 5. Theorem 8 and Corollary 9 do not hold, in general, if the cone $\widehat{N}_{Y_j}(\cdot)$ of limiting normal vectors are replaced by the cone of perpendicular vectors $\perp_{Y_j}(\cdot)$. ■

4.4. Proof of the second welfare theorem

The proof of Theorem 8 relies on the following result, which was used in a similar context by Bonnisseau [28].

LEMMA 2 (Cornet–Rockafellar [49]). *Let C_k $(k \in K)$ be a finite family of subsets of R^ℓ, let $\bar{c}_k \in \operatorname{cl} C_k$ $(k \in K)$ and let $\varepsilon > 0$. Then*

$$\sum_{k\in K} \bar{c}_k \in \partial\Big[\sum_{k\in K} \operatorname{cl} C_k \cap \overline{B}(\bar{c}_k,\varepsilon)\Big] \implies \bigcap_{k\in K} \hat{N}_{C_k}(\bar{c}_k) \neq \{0\}.$$

Proof. Step 1. We first prove the lemma under the additional assumption that the sets C_k are closed and that the following stronger condition holds: $\sum_k \bar{c}_k \in \partial[\sum_k C_k]$. From this condition we deduce the existence of a sequence $(e^\nu) \subseteq R^\ell$ converging to zero such that, for all ν we have:

$$-e^\nu + \sum_k \bar{c}_k \notin \sum_k C_k.$$

We consider the following minimization problem

$$(P^\nu) \quad \text{Minimize} \quad \Big\| e^\nu + \sum_{k\in K}(x_k - \bar{c}_k)\Big\| + \sum_{k\in K}\Big\| x_k - \bar{c}_k\Big\|^2$$
$$\text{Subject to} \quad x_k \in C_k \quad (k \in K),$$

where $x_k \in \mathbb{R}^\ell$ $(k \in K)$ are the variables of the problem (P^ν) and e^ν and $\bar{c}_k$ $(k \in K)$ are fixed parameters.

Claim 1. For every ν, the problem (P^ν) admits a solution (not necessarily unique), denoted by $(x_k^\nu)_{k\in K}$, and the sequence (x_k^ν) converges to $\bar{c}_k$ for every k. Indeed, first the existence of the solution is a clear consequence of the fact that the criterium of the problem is coercive and the sets C_k ($k \in K$) are closed. We now denote by v^ν the value of the problem (P^ν), that is,

$$v^\nu := \left\| e^\nu + \sum_{k\in K} x_k^\nu - \bar{c}_k \right\| + \sum_{k\in K} \left\| x_k^\nu - \bar{c}_k \right\|^2.$$

Consequently, for every ν and every k one has $\|x_k^\nu - \bar{c}_k\|^2 \leq v^\nu$. Moreover, letting $x_k = \bar{c}_k$ for all k in the problem (P^ν), one gets $v^\nu \leq \|e^\nu\|$, which we recall converges to zero . Thus (x_k^ν) converges to $\bar{c}_k$. This ends the proof of the claim.

For all ν, the solution $(x_k^\nu)_{k\in K}$ of (P^ν) satisfies the first order necessary conditions of the problem (P^ν), which can be written as follows [see Proposition 6.2]

$$-p^\nu - 2(x_k^\nu - \bar{c}_k) \in \perp_{C_k} (x_k^\nu) \text{ for all } k \text{ and } \nu, \tag{$*$}$$

where $p^\nu = e^\nu + \sum_{k\in K}(x_k^\nu - \bar{c}_k)/\|e^\nu + \sum_{k\in K}(x_k^\nu - \bar{c}_k)\|$ is well defined, recalling that

$$-e^\nu + \sum_k \bar{c}_k \neq \sum_{k\in K} x_k^\nu \in \sum_k C_k.$$

Without any loss of generality, we can suppose that the sequence (p^ν) in the unit sphere S, converges to some element $\bar{p}$ in S (hence is non-zero). Taking the limit in ($*$), when $\nu \to \infty$, for all k, one gets $-\bar{p} = -\lim_\nu p^\nu \in \hat{N}_{C_k}(\bar{c}_k)$ since $\bar{c}_k = \lim_\nu x_k^\nu$ (from Claim 1). This ends the proof of Step 1. ■

Step 2. We now present the proof of the lemma in the general case. Letting $D_k = \operatorname{cl} C_k \cap \overline{B}(x, \varepsilon)$. From Step 1, we deduce that $\bigcap_{k\in K} \hat{N}_{D_k}(\bar{c}_k) \neq \{0\}$. But, from Proposition 6.2, one gets

$$\hat{N}_{D_k}(\bar{c}_k) = \hat{N}_{D_k \cap B(\bar{c}_k,\varepsilon)}(\bar{c}_k) = \hat{N}_{\operatorname{cl} C \cap B(\bar{c}_k,\varepsilon)}(\bar{c}_k) = \hat{N}_{\operatorname{cl} C}(\bar{c}_k) = \hat{N}_{C_k}(\bar{c}_k),$$

hence the desired conclusion follows. ■

We come back to the proof of Theorem 8, which is a consequence of Lemma 2 and the two following claims.

CLAIM 1. *There exist $e \in \mathbb{R}^\ell$ and $\varepsilon > 0$ such that, for every $t \in (0, \varepsilon)$,*

$$te + \sum_{i\in I} \operatorname{cl} P_i(x_i) \cap \overline{B}(x_i, \varepsilon) - \sum_{j\in J} \operatorname{cl} Y_j \cap \overline{B}(y_j, \varepsilon) \subseteq \sum_{i\in I} P_i(x_i) - \sum_{j\in J} Y_j.$$

Hence, there is a sequence $e^\nu \to 0$ such that, for all ν we have

$$e^\nu + \sum_{i\in I} \operatorname{cl} P_i(x_i) \cap \overline{B}(x_i, \varepsilon) - \sum_{j\in J} \operatorname{cl} Y_j \cap \overline{B}(y_j, \varepsilon) \subseteq \sum_{i\in I} P_i(x_i) - \sum_{j\in J} Y_j.$$

Proof of Claim 1. Since Y_j is closed, it is sufficient to show that, for every i there exist e_i and $\varepsilon_i > 0$ such that $t \in (0, \varepsilon_i)$ implies:

$$te_i + \operatorname{cl} P_i(x_i) \cap \overline{B}(x_i, \varepsilon) \subseteq P_i(x_i). \tag{$**$}$$

Then $e = \sum_{i \in I} e_i$ and $\varepsilon = \min_i \varepsilon_i$ will satisfy the desired conclusion.

We now show that the set $C := P_i(x_i)$, which is convex by assumption, satisfies the above inclusion $(**)$. For this, we chose $\bar{c} \in \operatorname{ri} C$, the relative interior of the convex set C, and we let $e_i := \bar{c} - x_i$ and $\varepsilon \in (0, 1)$ be such that $A \cap \bar{B}(\bar{c}, \varepsilon) \subseteq \operatorname{ri} C$, (where A is the affine space spanned by C) and we let $c \in \operatorname{cl} C \cap \bar{B}(x_i, \varepsilon)$. Then $te_i + c = t(\bar{c} - x_i + c) + (1 - t)c$ with $\bar{c} - x_i + c \in A \cap \bar{B}(\bar{c}, \varepsilon) \subseteq \operatorname{ri} C$, and $c \in \operatorname{cl} C$. Consequently, $te_i + c$ belongs to $\operatorname{ri} C \subseteq C$; see, for example, Rockafellar [100]. ■

CLAIM 2. $\omega = \sum_{i \in I} x_i - \sum_{j \in J} y_j$ *and*

$$\omega \in \partial \Big[\sum_{i \in I} \operatorname{cl} P_i(x_i) \cap \overline{B}(x_i, \varepsilon) - \sum_{j \in J} \operatorname{cl} Y_j \cap \overline{B}(y_j, \varepsilon) \Big].$$

Proof of Claim 2. Let (e^ν) be the sequence converging to 0 given by Claim1. It suffices to show that for every ν one has

$$\omega - e^\nu \notin \sum_{i \in I} \operatorname{cl} P_i(x_i) \cap \overline{B}(x_i, \varepsilon) - \sum_{j \in J} \operatorname{cl} Y_j \cap \overline{B}(y_j, \varepsilon).$$

Suppose it is not true. Then, by Claim 1, there exists some ν such that

$$\begin{aligned} \omega \ &\in \ e^\nu + \sum_{i \in I} \operatorname{cl} P_i(x_i) \cap \overline{B}(x_i, \varepsilon) - \sum_{j \in J} \operatorname{cl} Y_j \cap \overline{B}(y_j, \varepsilon) \\ &\subseteq \ \sum_{i \in I} P_i(x_i) - \sum_{j \in J} Y_j, \end{aligned}$$

which contradicts the fact that $((x_i), (y_j))$ is a weakly Pareto optimal allocation. ■

REMARK 6. The above Theorem 8 was proved in Cornet [44] by showing that the following maximization problem

$$\begin{array}{rl} \text{Maximize} & t \\ \text{Subject to:} & \sum_{i \in I} x'_i - \sum_{j \in J} y'_j - \omega + te = 0, \\ & t \in (-\varepsilon, +\varepsilon), \\ & x'_i \in \operatorname{cl} P_i(x_i) \cap B(x_i, \varepsilon), \ (i \in I) \\ & y'_j \in \operatorname{cl} Y_j \cap B(y_j, \varepsilon), \quad (j \in J) \end{array}$$

admits for solution the point $(0,(x_i),(y_j,))$. Then, it was shown that the Lagrange multiplier $p \in \mathbb{R}^\ell$ associated to the equality constraint (using the Lagrangian multiplier rule of Clarke [43, 6.1.1, and 6.1.2 (iv)]), satisfies the conclusion of Theorem 8, when the marginal pricing rule is defined by Clarke's normal cone, that is, the closed convex hull of the cone of limiting normal vectors. The convexity of Clarke's normal cone was shown to be fundamental in the existence problem of marginal cost pricing equilibria [30, 31, 32, 33, 46].

The same maximization problem is considered by Khan [70], a paper that has been circulating since 1987, to prove that, in Theorem 8, Clarke's normal cone can be replaced by the smaller cone of limiting normal vectors, using the Lagrangian multiplier rule by Mordukhovicz [90]. The earlier work by Guesnerie [65] was using another cone, namely the Dubovickii–Miljutin's one, with a dual approach via the tangent cone (see also, in the infinite dimensional setting, Bonnisseau–Cornet [29]). Generalizations of the above Lemma 2 to the infinite dimensional setting are considered by [34, 91].

4.5. The general case

The proof of the Second Welfare Theorem that we gave in the previous section works for a much larger class of economies. This leads us to introduce the following Constraint Qualification Assumption that was introduced in [44]. As before we consider the economy:

$$\mathcal{E} = (\mathbb{R}^\ell, (X_i, \omega_i \succeq_i)_{i\in I}, (Y_j)_{j\in J})\,.$$

DEFINITION 6. *The allocation $((x_i),(y_j))$ of the economy $\mathcal{E}$ is said to be qualified if, for every $i \in I$, $x_i \in \operatorname{cl} P_i(x_i)$ (which holds in particular under Local Nonsatiation), and there exist $\varepsilon > 0$ and a sequence $(e^\nu) \subseteq \mathbb{R}^\ell$ converging to zero such that for every ν we have*

$$e^\nu + \sum_{i\in I} \operatorname{cl} P_i(x_i) \cap \overline{B}(x_i,\varepsilon) - \sum_{j\in J} \operatorname{cl} Y_j \cap \overline{B}(y_j,\varepsilon) \subseteq \sum_{i\in I} P_i(x_i) - \sum_{j\in J} Y_j.$$

We now state a general version of the Second Welfare Theorem, in which no convexity, no differentiability, and no interiority assumption are made, neither on the production sets, nor on the consumers' characteristics.

THEOREM 10. *Let $((x_i),(y_j))$ be a weakly Pareto optimal allocation of the economy $\mathcal{E}$, and assume that it is qualified. Then, there exists a non-zero price p in $\mathbb{R}^\ell$ such that:*

- *For each consumer i, $-p \in \widehat{N}_{P_i(x_i)}(x_i)$.*
- *For each producer j, $p \in \widehat{N}_{Y_j}(y_j)$.*

The proof of Theorem 10 is exactly the same as the one given in the previous section, with the only difference that we don't have to prove Claim 1, which is exactly the Qualification Assumption of the economy. We refer to Cornet [44] and Khan [71] for a more detailed discussion on the relationship with other results on this subject. We note again that Theorem 10 does not hold, in general, if the cones $\widehat{N}_{P_i(x_i)}(\cdot)$ and $\widehat{N}_{Y_j}(\cdot)$ of limiting normal vectors are replaced by the smaller cones of perpendicular vectors $\perp_{P_i(x_i)}$ and $\perp_{Y_j}$.

We end this section by giving several cases of economic interest under which an allocation is qualified.

PROPOSITION 11. *(a) The allocation $((x_i),(y_j))$ of the economy $\mathcal{E}$ is qualified if, for* **every** *$i \in I$, the couple $(C,c) = (P_i(x_i), x_i)$* **and***, for* **every** *$j \in J$ the couple $(C,c) = (Y_j, y_j)$ satisfies one of the following conditions:*

(i) there exist $\varepsilon > 0$ and a sequence (e^ν) converging to 0, such that

$$\forall \nu, \quad e^\nu + \operatorname{cl} C \cap B(c,\varepsilon) \subseteq C.$$

(ii) [Convexity] C is convex;

(iii) $\operatorname{cl} C + Q \subseteq C$, for some closed cone $Q \subseteq \mathbb{R}^\ell$ (which is satisfied, in particular, when C is closed, with $Q = \{0\}$) .

(b) The allocation $((x_i),(y_j))$ of the economy $\mathcal{E}$ is qualified if, for **some** *$i \in I$, the couple $(C,c) = (P_i(x_i), x_i)$* **or***, for* **some** *$j \in J$ the couple $(C,c) = (Y_j, y_j)$ satisfies one of the following conditions:*

(i′) there exist $\varepsilon > 0$ and a sequence (e^ν) converging to 0, such that

$$\forall \nu, \quad e^\nu + \operatorname{cl} C \cap B(c,\varepsilon) \subseteq \operatorname{int} C.$$

(ii′) C is convex, with a nonempty interior;

(iii′) [Free Disposal] $C + Q \subseteq C$, for some closed cone $Q \subseteq \mathbb{R}^\ell$, with a nonempty interior.

The proof of the proposition is a consequence of the two following Claims.

CLAIM 3. *Condition (i) (resp. (i')) is satisfied by $C \subseteq \mathbb{R}^\ell$, and $c \in \operatorname{cl} C$, if one of the conditions (ii) or (iii) (resp. (ii') or (iii')) holds.*

Proof. $(ii) \implies (i)$. It is shown in the proof of Claim 1.
$(iii) \implies (i)$. Take $e \in Q$.
$(ii') \implies (i')$. The set $\operatorname{int} C$ is convex, and $\operatorname{cl} C = \operatorname{cl} \operatorname{int} C$ (see, for example [100]). Hence, the implication $[(ii) \implies (i)]$ (applied to $\operatorname{int} C$), gives us the conclusion.

$(iii') \implies (i')$. We prove that every $e \in \operatorname{int} Q$ satisfies the stronger condition that $te + \operatorname{cl} C \subseteq C$ for every $t > 0$. Indeed, if $c \in \operatorname{cl} C$, there exists a sequence $(c_n) \subseteq C$ converging to c. Hence, for every $t > 0$, $te + c = t[e + (c - c_n)/t] + c_n$. But, for n large enough, the point $e_n := e + (c - c_n)/t \in \operatorname{int} Q$ and from the Free Disposal Assumption, we get $c + te = c_n + te_n \in C + \operatorname{int} Q \subseteq C$. ■

CLAIM 4. *Let $C_k \subseteq \mathbb{R}^\ell$ ($k \in K$) be a finite family, let $c_k \in \operatorname{cl} C_k$, and assume that the couple (C_k, c_k) satisfies Condition (i) for* **every** *k (resp. Condition (i') for* **some** *k), then there exist $\varepsilon > 0$ and a sequence (e^ν), converging to 0, such that*

$$(Q) \qquad \forall \nu, \quad e^\nu + \sum_{k \in K} \operatorname{cl} C_k \cap B(c_k, \varepsilon) \subseteq \sum_{k \in K} C_k.$$

Proof of Claim 4. Let us first assume that the couple (C_k, c_k) satisfies Condition (i) for **every** k. Then, there exist $\varepsilon_k > 0$ ($k \in K$) and a sequence (e_k^ν) ($k \in K$), converging to 0, such that

$$\forall \nu, \quad e_k^\nu + \operatorname{cl} C_k \cap B(c_k, \varepsilon_k) \subseteq C_k.$$

We define the sequence $e^\nu := \sum_{k \in K} e_k^\nu$ and $\varepsilon := \min_k \varepsilon_k$. Taking the sum of the above inclusions, we get

$$e^\nu + \sum_{k \in K} \operatorname{cl} C_k \cap B(c_k, \varepsilon) \subseteq \sum_{k \in K} C_k.$$

We now consider the case where the couple (C_k, c_k) satisfies Condition (i') for **some** k, say $k = 1$. Then there exist (ε_1) and a sequence (e_1^ν), converging to 0, such that

$$\forall \nu, \quad e_1^\nu + \operatorname{cl} C_1 \cap B(c_1, \varepsilon_1) \subseteq \operatorname{int} C_1.$$

Consequently, for every ν

$$e_1^\nu + \sum_{k \in K} \operatorname{cl} C_k \cap B(c_k, \varepsilon_1) \subseteq \operatorname{int} C_1 + \sum_{k \neq 1} \operatorname{cl} C_k,$$

and one checks that $\operatorname{int} C_1 + \sum_{k \neq 1} \operatorname{cl} C_k \subseteq \sum_{k \in K} C_k$. ■

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