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Existence of generalized equilibria

Bernard Cornet[a,*], Marc-Olivier Czarnecki[b]

[a] *CERMSEM, Université de Paris 1, 106-112 Boulevard de l'Hopital, 1 Pantheon-Sorbonne, 90 rue de Tolbiac, 75013 Paris, France*

[b] *Laboratoire d'Analyse Convexe, Université de Montpellier 2, place Eugène Bataillon, 34095 Montpellier Cedex 5, France*

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1. Introduction[1]

We consider the problem of the existence of equilibria, generalized equilibria, and fixed-points of a correspondence F, defined on a compact subset M of $\mathbb{R}^n$, with values in $\mathbb{R}^n$, when the set M is neither assumed to be convex, nor smooth. [1]

The general framework of the article is the following. Let F be a correspondence from M to $\mathbb{R}^n$, that is, a map from M to the set of all the subsets of $\mathbb{R}^n$; the correspondence F is said to be upper semicontinuous (u.s.c.), if the set $\{x \in M \mid F(x) \subset V\}$ is open in M for every open set $V \subset \mathbb{R}^n$. Let N be a "normal cone" correspondence, i.e., at this stage, N is a correspondence from M to $\mathbb{R}^n$ such that, for every $x \in M$, $N(x)$

* Corresponding author. Tel.: +33-404631-66; fax: +33-4051-8064.

E-mail addresses: cornet@univ-paris1.fr (B. Cornet), marco@math.univ-montp2.fr (M. Czarnecki).

[1] We let $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ and sgn $x = x/|x|$ if $x \in \mathbb{R} \setminus \{0\}$. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ belong to $\mathbb{R}^n$, we denote $(x \mid y) = \sum_{i=1}^n x_i y_i$, the scalar product of $\mathbb{R}^n$, $\|x\| = \sqrt{(x \mid x)}$, the Euclidean norm; we denote $B(x,r) = \{y \in \mathbb{R}^n \mid \|x - y\| < r\}$, $\bar{B}(x,r) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$ and $S(x,(r)) = \{y \in \mathbb{R}^n \mid \|x - y\| = r\}$. If $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^n$, and $x \in \mathbb{R}^n$, we let $d_X(x) = \inf_{y \in X} \|x - y\|$, $X \setminus Y = \{x \in X \mid x \notin Y\}$ the set-difference of the sets X and Y, $X + Y = \{x + y \mid x \in X, y \in Y\}$, the sum of the sets X and Y, $B(X,r) = X + B(0,r)$, $\bar{B}(X,r) = X + \bar{B}(0,r)$, $\mathrm{cl}\, X$, the closure of X, int X, the interior of X, $\mathrm{bd}\, X = \mathrm{cl}\, X \setminus \mathrm{int}\, X$, the boundary of X, $X^\circ = \{y \in \mathbb{R}^n \mid \forall x \in X,\ (y \mid x) \leq 0\}$, the negative polar cone of X, co X, the convex hull of X. A map $f : X \to \mathbb{R}$ is locally Lipschitzian if, for every $x \in X$, there is $\varepsilon > 0$ and $L > 0$ such that $\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\|$ for every x_1 and x_2 in $B(x, \varepsilon)$. If F is a correspondence from X to $\mathbb{R}^n$, its graph, denoted by $G(F)$, is defined by $G(F) = \{(x,y) \in X \times \mathbb{R}^n \mid y \in F(x)\}$.

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is a cone (not necessarily convex) and, in the following, we shall take for N one of the notions of normal cones available in the literature (see below).

Our main result considers the problem of the existence of generalized equilibria. Formally, we provide sufficient conditions for the following assertion to hold:

Assertion $(GE;N)$ (Generalized equilibria)**.** *For every u.s.c. correspondence F from M to $\mathbb{R}^n$, with nonempty, convex, compact values, there exists a generalized equilibrium $x^* \in M$ of F, i.e.*

$$x^* \in M \quad \textit{such that } 0 \in F(x^*) - N(x^*).$$

In this paper, the cone N is chosen to be either Clarke's normal cone, denoted by N_M, or a smaller cone defined hereafter, which allows us to get finer existence results. We shall also consider the case of other normal cones, such as Bouligand normal cone and the limiting normal cone, but they will appear not to be adapted to the above existence problem (see Section 5). We shall also relate the above problem with the more usual one which considers the existence of equilibria under an additional tangential condition, as formulated in the following assertion. We denote $N(x)^\circ$ the negative polar cone of $N(x)$, and we recall that, if $N = N_M$, $N(x)^\circ$ is Clarke's tangent cone to M at x.

Assertion $(E;N)$ (Equilibria)**.** *For every u.s.c. correspondence F from M to $\mathbb{R}^n$, with nonempty, convex, compact values, such that $F(x) \cap N(x)^\circ \neq \emptyset$ for every $x \in M$, there exists an equilibrium $x^* \in M$ of F, i.e.*

$$x^* \in M \quad \textit{such that } 0 \in F(x^*).$$

Assertion $(E;N)$ can be equivalently reformulated in terms of the existence of fixed points as follows.

Assertion $(FP;N)$ (Fixed points)**.** *For every u.s.c. correspondence F from M to $\mathbb{R}^n$, with nonempty, convex, compact values, such that $F(x) \cap (\{x\} + N(x)^\circ) \neq \emptyset$ for every $x \in M$, there exists a fixed point $x^* \in M$ of F, i.e.*

$$x^* \in M \quad \textit{such that } x^* \in F(x^*).$$

Before stating our main result, we recall some definitions. If $X \subset \mathbb{R}^n$, and if $\bar{x} \in \mathbb{R}^n$ we define

$$\limsup_{x \to \bar{x}, x \in X} F(x) = \{v \in \mathbb{R}^n \mid \exists (x_k) \subset X,\ \exists (v_k) \subset \mathbb{R}^n, x_k \to \bar{x},\ v_k \in F(x_k), v_k \to v\}.$$

Let U be an open subset of $\mathbb{R}^n$. If $f : U \to \mathbb{R}$ is differentiable at $x \in U$, we denote by $\nabla f(x)$ the gradient of f at x. If $f : U \to \mathbb{R}$ is locally Lipschitzian and $\bar{x} \in U$, we define its subdifferential at $\bar{x}$, denoted $\partial f(\bar{x})$, the set $(\hat{\nabla}_+ f(\bar{x}))$, and the set $\partial_+ f(\bar{x})$ by

$$\partial f(\bar{x}) = \text{co} \limsup_{x \to \bar{x}, x \in \text{Dom}(\nabla f)} \{\nabla f(x)\},$$

$$\nabla_+ f(\bar{x}) = \limsup_{x\to\bar{x}, x\in \mathrm{Dom}(\nabla f),\ f(x)>f(\bar{x})} \{\nabla f(x)\},$$

$$\partial_+ f(\bar{x}) = \limsup_{x\to\bar{x},\, f(x)>f(\bar{x})} \partial f(x),$$

where $\mathrm{Dom}(\nabla f)$ is the set on which f is differentiable. [2]

Let $M \subset \mathbb{R}^n$ be closed. Since d_M, the distance function to M, is Lipschitzian, from above, one can define, at $x \in M$, the limiting normal cone $\hat{N}_M(x)$, the cone $\tilde{N}_M(x)$, and Clarke's normal cone $N_M(x)$ by

$$\hat{N}_M(x) = \bigcup_{\lambda\geq 0} \lambda \nabla_+ d_M(x) \cup \{0\},$$

$$\tilde{N}_M(x) = \bigcup_{\lambda\geq 0} \lambda \partial_+ d_M(x) \cup \{0\},$$

$$N_M(x) = \mathrm{cl}\left(\bigcup_{\lambda\geq 0} \lambda \partial d_M(x) \right).$$

We now state the main result of the paper.

Theorem A. *If M is a nonempty, compact subset of $\mathbb{R}^n$ such that*

$$0 \notin \partial_+ d_M(x) \quad \text{for every } x \in M, \tag{1}$$

then M has a finite number of connected components $(M^i)_{i\in I}$, each of which has a well-defined Euler characteristic denoted $\chi(M^i)$.

Furthermore, each of the following assertions $(GE;\tilde{N}_M)$, $(GE;N_M)$, $(E;\tilde{N}_M)$, and $(E;N_M)$ is satisfied if:

Assertion (χ). *There is a connected component M^i of M such that $\chi(M^i) \neq 0$.*

A set $M \subset \mathbb{R}^n$ satisfying Condition (1) is said to be proximally nondegenerate. It is worth pointing out that the class, denoted as $\mathscr{M}$, of proximally nondegenerate subsets of $\mathbb{R}^n$ contains, in particular, the classes of convex subsets of $\mathbb{R}^n$, differentiable submanifolds of $\mathbb{R}^n$, epi-Lipschitzian subsets of $\mathbb{R}^n$ in the sense of Rockafellar [25]. The main properties of proximally nondegenerate sets are given in Section 2. In Section 5, we shall show that $\mathscr{M}$ contains the classes of proximally regular subsets of $\mathbb{R}^n$ [23,24], proximally smooth subsets of $\mathbb{R}^n$ [8,18], and that $\mathscr{M}$ is contained in the class of $\mathscr{L}$-retracts subsets of $\mathbb{R}^n$, in the sense of Ben-El-Mechaiekh and Kryszewski [2].

Assertion (χ) is a topological assumption on the set M which is satisfied if one of the following holds: (i) $\chi(M) \neq 0$ (since one shows that $\chi(M) = \sum_i \chi(M^i)$), (ii) M is convex (since $\chi(M) = 1$), (iii) M is homeomorphic to a convex set (since the Euler characteristic is invariant by homeomorphism). We shall also show (Theorem 2.3) that Assertion (χ) is equivalent to each of the assertions $(GE;\tilde{N}_M)$, $(GE;N_M)$, $(E;\tilde{N}_M)$, and $(E;N_M)$ of Theorem A if we additionally assume that the set M is epi-Lipschitzian.

[2] We recall that $U \setminus \mathrm{Dom}(\nabla f)$ is of Lebesgue measure zero, from Rademacher's theorem.

This result, which extends a previous one by Cornet and Czarnecki [14], may not be true if M is only proximally nondegenerate (see Remark 1, Section 2.3).

We now discuss the link between our main result (Theorem A) and previous ones in the literature. Theorem A generalizes known results in finite dimension, in the convex and in the smooth cases. If M is convex and $x \in M$, $N_M(x)^\circ = \tilde{N}_M(x)^\circ$ is the tangent cone in the sense of convex analysis, thus the implication $[\chi(M) \neq 0 \Rightarrow (FP; N_M)]$ is the existence theorem of fixed points for inward (or outward) correspondences proved in a long tradition of authors in the more general setting of infinite-dimensional spaces (for example [3,9]), which also generalizes the classical results of Brouwer and Kakutani. If M is smooth (i.e., a C^2 submanifold with a boundary of $\mathbb{R}^n$, of full dimension), the implication $[(\chi(M) \neq 0) \Rightarrow (E; N_M)]$, in the single-valued case, is essentially the existence part of Poincaré-Hopf's theorem (for example, see [22]). The equivalence $[(\chi(M) \neq 0) \Leftrightarrow (E; N_M)]$ is well known when M is additionally assumed to be connected and the correspondence F to be single-valued (see for example [27]).

In the nonsmooth and nonconvex case, the existence problem of equilibria has been considered by several authors [11,7,2,14]). In [11,14], the implication $[(\chi(M) \neq 0) \Rightarrow (E; N_M)]$ is proved if M is nonempty, compact, and epi-Lipschitzian, using degree theory in [11], and approximation methods in [14]. In [7], it is shown that Assertion $(E; N_M)$ holds if M is compact, epi-Lipschitzian and homeomorphic to a convex set (this condition implies that $\chi(M) = 1$) in $\mathbb{R}^n$ and, possibly, in an infinite-dimensional setting using techniques from differential equations and fixed-point theorems (Brouwer, Kakutani). Finally, [2] shows the implication $[\chi(M) \neq 0 \Rightarrow (E; N_M)]$ for the class of $\mathscr{L}$-retract sets in an infinite-dimensional setting, using Lefschetz fixed-point theorem. The main novelty of our main result (Theorem A) is more on the existence of generalized equilibria (see below), since the implication $[(\chi(M) \neq 0) \Rightarrow (E; N_M)]$ is proved in [2], in a more general setting, but our proof uses different techniques from approximation theory. As a by-product, also of interest for itself, we introduce a subclass of $\mathscr{L}$-retract sets, the class of proximally nondegenerate sets, defined by a geometric and intrinsic condition (Condition (1)) which is easy to check and work with. [3]

The existence of generalized equilibria is quite standard in the convex case, in which the generalized equation $0 \in F(x) - N_M(x)$ is called variational inequality. In the nonconvex case, [11,14] consider the epi-Lipschitzian case and prove the implication $[(\chi(M) \neq 0) \Rightarrow (GE; N_M)]$. Conversely, [14] also shows that Assertion $(GE; N_M)$ is equivalent to Assertion (χ). Our main result (the implication $[(\chi) \Rightarrow (GE; \tilde{N}_M)]$ of Theorem A) extends the previous existence result in two ways. First, by enlarging the class of epi-Lipschitzian sets to the class of proximally nondegenerate sets. Second, by considering Assertion $(GE; \tilde{N}_M)$, which always implies $(GE; N_M)$ and may not be equivalent to it (see Remark 3, Section 2.2), and thus provides more precise results than the implication $[(\chi) \Rightarrow (GE; N_M)]$ with Clarke's normal cone. Finally, it is worth pointing out that our existence result of equilibria is deduced from our existence result of generalized equilibria, which thus provides a more general setting to study the existence problem.

[3] After the paper was submitted, we received a manuscript from Ćwiszewski and Kryszewski [16], which shows the implication $[(\chi(\mathscr{M}) \neq 0) \Rightarrow (GE; N_M)]$ for the class of $\mathscr{L}$-retract sets. They also consider the class of strictly regular sets, also related to the class $\mathscr{M}$.

The paper is organized as follows. In Section 2, we state the main properties of proximally nondegenerate sets and show that they admit a smooth normal approximation (Theorem 2.1) in a precise sense given hereafter. Then we present a generalization of Theorem A by considering the class of sets admitting a smooth normal approximation. In Section 3, we give the proof of the existence result. In Section 4, we give the proof of the approximation result (Theorem 2.1). Finally, in Section 5, we consider the existence problem with other normal cones such as Bouligand normal cone and the limiting normal cone, and we show the relation between proximally nondegenerate sets and other classes of sets previously used in the literature (proximally regular sets and proximally smooth sets, $\mathscr{L}$-retract, Lipschitzian submanifolds).

2. Statement of the results

The following proposition summarizes the main properties used in the following (see [6,15] for the proof).

Proposition 2.1. *Let $f:\mathbb{R}^n \to \mathbb{R}$ be locally Lipschitzian, and let M be a closed subset of $\mathbb{R}^n$. Then*:

(a) $\nabla_+ f(x) \subset \partial_+ f(x) \subset \operatorname{co} \nabla_+ f(x) = \operatorname{co} \partial_+ f(x) \subset \partial f(x)$, *for every* $x \in \mathbb{R}^n$ (*and each inclusion may be strict*);

(b) *the correspondence $x \mapsto \partial f(x)$ is u.s.c., with nonempty convex compact values*;

(c) *the correspondence $x \mapsto \partial_+ f(x)$ is u.s.c., with compact values* (*not necessarily convex*);

(d) *the set $\partial_+ f(x)$ is nonempty if and only if x is not a local maximum of f*;

(e) *if f is of class C^1 on a neighborhood of x, then $\partial_+ f(x) \subset \partial f(x) = \{\nabla f(x)\}$*;

(f) $\hat{N}_M(x) \subset \tilde{N}_M(x) \subset \operatorname{clco} \hat{N}_M(x) = \operatorname{clco} \tilde{N}_M(x) = N_M(x)$, *for every* $x \in M$; [4]

(g) $\hat{N}_M(x) \cap S = \nabla_+ d_M(x)$, *for every* $x \in \operatorname{bd} M$.

2.1. Existence of smooth normal approximations

In this section, we prove a fundamental approximation property of the sets M in the class $\mathscr{M}$, which we recall is defined as follows.

Definition 2.1. A closed subset M of R^n is proximally nondegenerate at $x \in M$, if one of the following equivalent assertions is satisfied:

(i) $0 \notin \partial_+ d_M(x)$;

(ii) $\exists \alpha > 0,\ \forall x' \in B(x, \alpha) \setminus M,\ \forall p \in \partial d_M(x'),\ \|p\| \geq \alpha$.

The set M is said to be proximally nondegenerate if it is proximally nondegenerate at every $x \in M$. We denote M the set of proximally nondegenerate closed subsets of $\mathbb{R}^n$.

[4] Moreover, in view of Clarke [6], $\hat{N}_M(x) = \limsup_{x' \to x, x' \in M} \perp_M(x')$, where $\perp_M(x) = \{v \in \mathbb{R}^n \mid \exists \alpha > 0, B(x + \alpha v, \alpha\|v\|) \cap M = \emptyset\}$; this a more usual way to define the limiting normal cone $\hat{N}_M(x)$.

Remark. One could think of replacing, in (i), the set $\partial_+ d_M(x)$ by $\nabla_+ d_M(x)$. In this case, every subset of $\mathbb{R}^n$ satisfies this new assumption, since, $\|\nabla d_M(x)\| = 1$ if d_M is differentiable at x, hence $0 \notin \nabla_+ d_M(x) \subset S$ for every $x \in M$.

Also, one could think of enlarging the class $\mathscr{M}$ by considering the class $\mathscr{M}'$ of closed subsets M of $\mathbb{R}^n$ such that:

$$\forall x \in M, \exists \alpha > 0, \forall x' \in B(x,\alpha)\backslash M, 0 \notin \partial d_M(x').$$

As we shall see below (Remark 5, Section 2.2), our main existence result (Theorem A in the introduction) may not hold if the set M is only assumed to belong to the class $\mathscr{M}'$. We shall also see (Remark 4 below) that sets in $\mathscr{M}'$ have less topological properties that sets in $\mathscr{M}$.

The class $\mathscr{M}$ contains in particular closed convex subsets of $\mathbb{R}^n$, closed differentiable submanifolds of $\mathbb{R}^n$, closed epi-Lipschitzian subsets of $\mathbb{R}^n$ as shown by the two following propositions (see [15] for their proofs):

Proposition 2.2. *Let M be a closed subset of $\mathbb{R}^n$. Then the two following conditions are equivalent*:

(i) *M is epi-Lipschitzian* (*i.e.*, $N_M(x) \cap -N_M(x) = \{0\}$ *for every* $x \in M$).

(ii) $0 \notin \operatorname{co} \partial_+ d_M(x)$, *for every* $x \in M$.

Proposition 2.3. *The class $\mathscr{M}$ contains the closed sets $M \subset \mathbb{R}^n$ which satisfy one of the following conditions*:

(i) *M is convex*;

(ii) *$\mathbb{R}^n \backslash M$ is convex*;

(iii) *M is a C^1 submanifold of $\mathbb{R}^n$, with or without a boundary, with or without corners*;

(iv) *M is epi-Lipschitzian.*

Note that every closed convex subset of $\mathbb{R}^n$ belongs to $\mathscr{M}$, but may not be epi-Lipschitzian if it has an empty interior. In the last section we shall further show that $\mathscr{M}$ contains proximally smooth sets, is contained in the class of $\mathscr{L}$-retracts sets, and that it is neither contained nor contains the class of Lipschitzian submanifolds of $\mathbb{R}^n$.

Our next results states that every element $M \in \mathscr{M}$ can be approximated in the following precise sense.

Theorem 2.1. *Let $M \in \mathscr{M}$ be a compact set. Then the correspondence $\tilde{N}_M$ has a closed graph and the set M admits a smooth normal approximation (M_k) in the sense that*

(i) *for every k, M_k is a compact and smooth subset of $\mathbb{R}^n$, i.e., is a closed C^∞ submanifold with a boundary of $\mathbb{R}^n$, of full dimension*;

(ii) *for every k, $M_{k+1} \subset M_k \subset B(M,1)$, and $M = \bigcap_{k\in\mathbb{N}} M_k$*;

(cn) $\limsup_{k\to\infty} \mathrm{G}(N_{M_k}) \subset \mathrm{G}(\tilde{N}_M)$;

(ret) *for every k, M is a deformation retract of M_k,*
i.e., there is a continuous map $H:[0,1] \times M_k \to M_k$ such that, for every $x \in M_k$, $H(0,x) = x$, and $H(1,x) \in M$, and, for every $x \in M$, $H(1,x) = x$.

The proof of Theorem 2.1 is given in Section 4.

Remark 3. It is worth pointing out that, if $M \in \mathscr{M}$, the correspondence N_M (Clarke's normal cone) may not have a closed graph. Consider in $\mathbb{R}^2$:

$$M = \bigcup_{n \in \mathbb{N}} \left[\left(\frac{1}{n+1}, \frac{1}{(n+1)^2} \right), \left(\frac{1}{n}, \frac{1}{n^2} \right) \right] \cup \{(0,0)\}.$$

Remark 4. Note from Assertion (ret) that every compact set M in $\mathscr{M}$ is a neighborhood retract. This is no longer true in general for sets in $\mathscr{M}'$ (see Remark 5 below).

2.2. *Existence of (generalized) equilibria*

In this section, we state our main result about the existence of equilibria and generalized equilibria in the case where $N = \tilde{N}_M$, and where $N = N_M$. Formally, we give sufficient conditions which imply Assertions $(GE; \tilde{N}_M)$, $(GE; N_M)$, $(E; \tilde{N}_M)$, and $(E; N_M)$ and their single-valued versions. [5]

We consider hereafter a larger class than $\mathscr{M}$, namely the class $\mathscr{A}$ of closed subsets of $\mathbb{R}^n$ which are approximable by smooth sets in the following sense:

Definition 2.2. We let $\mathscr{A}$ be the class of compact subsets M of $\mathbb{R}^n$ admitting a smooth normal approximation (M_k), i.e., which satisfy Assertions (i), (ii), (cn), and (ret) of Theorem 2.1.

If $M \in \mathscr{A}$ is a compact set, in view of Assertion (*ret*), we notice that its Euler characteristic $\chi(M)$ is well defined. We recall that

$$\chi(M) = \sum_{k \in \mathbb{N}} (-1)^k rg\, H_k M,$$

where $(H_k M)_{k \in \mathbb{N}}$ are the homology groups of M (see, for example [17]). At this stage, we only need to know that, if M belongs to $\mathscr{A}$, then each connected component M^i of M also belongs to $\mathscr{A}$ (see Proposition 3.1 below). [6] So Assertion (χ), given in the introduction, has a clear meaning:

Assertion (χ). *There is a connected component M^i of M such that $\chi(M^i) \neq 0$.*

Theorem 2.2. *Let $M \subset \mathbb{R}^n$ be a nonempty compact set which admits a smooth normal approximation, then*:

(a) $(\chi) \Rightarrow (GE; \tilde{N}_M) \Rightarrow (GE; N_M)$.

(b) *If we additionally assume that one of the correspondences $\tilde{N}_M$ or N_M has a closed graph, then*:

$$(\chi) \Rightarrow (E; \tilde{N}_M) \Leftrightarrow (E; N_M).$$

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[5] Respectively, denoted $(GE_{sv}; \tilde{N}_M)$, $(GE_{sv}; N_M)$, $(E_{sv}; \tilde{N}_M)$, and $(E_{sv}; N_M)$, in which we replace "u.s.c. correspondence F from M to $\mathbb{R}^n$, with nonempty, convex, compact values" by "continuous map $f: M \to \mathbb{R}^n$".

[6] We shall show that Assertion (ret) implies that M has a finite number of connected components $(M^i)_{i \in I}$ which belong to $\mathscr{A}$ and that $\chi(M) = \sum_{i \in I} \chi(M^i)$.

The proof of Theorem 2.2 is given in Section 3.

Remark 1. The equivalence $[(E;\tilde{N}_m) \Leftrightarrow (E;N_M)]$ is always true. Indeed, for every $x \in M$, $\mathrm{cl}(\mathrm{co}\,\tilde{N}_M(x)) = N_M(x)$ (Proposition 2.1), which implies that $\tilde{N}_M(x)^\circ = N_M(x)^\circ$.

Remark 2. The topological assumption (χ) is satisfied if $\chi(M) \neq 0$ (but the converse is not true in general if M is not connected). We recall that $\chi(M) = 1$ if M is convex.

In view of Theorems 2.1 and 2.2, one easily deduces the following corollary which is the result (Theorem A) stated in the introduction:

Corollary 2.1. *Let $M \in \mathscr{M}$, be nonempty and compact. Then Assertion (χ) implies each of the following assertions $(GE;\tilde{N}_M)$, $(GE;N_M)$, $(E;\tilde{N}_M)$, $(E;N_M)$, and their single-valued versions.*

Remark 3. The implication $[(GE;\tilde{N}_M) \Rightarrow (GE;N_M)]$ is always satisfied since $\tilde{N}_M(x) \subset N_M(x)$. However the converse, i.e., the implication $[(GE;N_M) \Rightarrow (GE;\tilde{N}_M)]$ may not be true. Consider the example of the square $M = \{(x,y) \in \mathbb{R}^2 \mid \sup\{|x|,|y|\} = 1\}$. So, Part (a) of Theorem 2.2 is strictly stronger than previous results in the literature which either consider the implication $[(\chi) \Rightarrow (GE;N_M)]$ ([11,12]) or the implication $[(\chi) \Rightarrow (E;N_M)]$ ([11,7,2,13]).

Remark 4. Each of the four implications of Theorem 2.2 ($[(\chi) \Rightarrow (GE;\tilde{N}_M)]$, $[(\chi) \Rightarrow (GE;N_M)]$, $[(\chi) \Rightarrow (E;\tilde{N}_M)]$, $[(\chi) \Rightarrow (E;N_M)]$) may no longer be true if M does not belong to $\mathscr{A}$. Consider the following connected set:

$$M = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{ll} & |x| \geq \frac{1}{4},\ (x - \frac{1}{4}\operatorname{sgn} x)^2 + (y - \operatorname{sgn} y)^2 \in [\frac{1}{2}, 1], \\ \text{or} & |x| < \frac{1}{4},\ y \in [-2, -\frac{3}{2}] \cup [-\frac{1}{2}, \frac{1}{2}] \cup [\frac{3}{2}, 2]. \end{array} \right.$$

Then M is compact, and $\chi(M) = -1$, hence Assertion (χ) holds true. Let $f : M \to \mathbb{R}^2$ be the map defined by $f(x,y) = (1 - |y|, (x - \operatorname{sgn} x/4)\operatorname{sgn} y)$ if $|x| > \frac{1}{4}$ and $f(x,y) = (1 - |y|, 0)$ if $|x| \leq \frac{1}{4}$. The map f is continuous on M, $f(x) \in T_M(x)$ (Clarke's tangent cone) for all $x \in M$, and $\varphi(x)$ is never equal to zero, hence Assertion $(E_{sv};N)$ holds false (hence the four assertions $(GE;\tilde{N}_M)$, $(GE;N_M)$, $(E;\tilde{N}_M)$ and $(E;N_M)$ hold false, since any of them clearly implies $(E_{sv};N)$).

Remark 5. Corollary 2.1 may not be true for sets in $\mathscr{M}'$. In $\mathbb{R}^2$, consider the spiral defined by:

$$M = S(0,1) \cup \{((1 + 1/\theta)\cos\theta, (1 + 1/\theta)\sin\theta) | \theta \in [3, \infty)\},$$

and the tangent field $f : M \to \mathbb{R}^2$ defined by $f((1 + 1/\theta)\cos\theta, (1 + 1/\theta)\sin\theta = (-\sin\theta - \cos\theta/\theta^2, \cos\theta - \sin\theta/\theta^2)$ and $f(\cos\theta, \sin\theta) = (-\sin\theta, \cos\theta)$. Then $M \in \mathscr{M}'$, $\chi(M) = 1$, and $f(x) \in T_M(x) \backslash \{0\}$ for every $x \in M$, hence none of the assertions $(GE;\tilde{N}_M)$, $(GE;N_M)$, $(E;\tilde{N}_M)$, $(E;N_M)$ is satisfied. Notice also that M is not a neighborhood retract.

2.3. The epi-Lipschitzian case: (χ) is necessary and sufficient

In this section, we investigate whether Assertion (χ) is also a necessary condition for the existence of (generalized) equilibria, i.e., formally, the four assertions of Theorem 2.2.

Our next result states that it is indeed the case when M is compact and epi-Lipschitzian.

Theorem 2.3. *Let M be a nonempty compact epi-Lipschitzian subset of $\mathbb{R}^n$. Then Assertion (χ) is equivalent to each of the following assertions $(GE;\tilde{N}_M)$, $(GE;N_M)$, $(E;\tilde{N}_M)$, $(E;N_M)$, and also equivalent to their single-valued versions.*

The above result extends a previous result from [14] which showed the equivalence between (χ), $(GE;N_M)$, $(E;N_M)$, and also the single-valued versions.

Proof of Theorem 2.3. The epi-Lipschitzian set M belongs to $\mathscr{M}$ from Proposition 2.3. Hence from Corollary 2.1, Assertion (χ) implies Assertions $(GE;\tilde{N}_M)$, $(GE;N_M)$, $(E;\tilde{N}_M)$, $(E;N_M)$ and their single-valued versions. Each of these eight assertions clearly implies Assertion $(E_{sv};N_M)$, and the implication $[(E_{sv};N_M) \Rightarrow (\chi)]$ is proved in [14]. □

Remark 1. Each of the four implications $[(GE;\tilde{N}_M) \Rightarrow (\chi)]$, $[(GE;N_M) \Rightarrow (\chi)]$, $[(E;\tilde{N}_M) \Rightarrow (\chi)]$ and $[(E;N_M) \Rightarrow (\chi)]$ of Theorem 2.3 is false in general, if the set M is not assumed to be epi-Lipschitzian, even if $M \in \mathscr{M}$. In $\mathbb{R}^2$, consider $M = S(0,1) \cup [0,2] \times \{0\}$ and note that $\chi(M) = 0$ and $\tilde{N}_M(1,0) = \mathbb{R}^2$. This example also shows that the converse of Theorem 2.2, part (a) is false in general.

Remark 2. Each of the three implications $[(GE;N_M) \Rightarrow (\chi)]$, $[(E;\tilde{N}_M) \Rightarrow (\chi)]$, and $[(E;N_M) \Rightarrow (\chi)]$ of Theorem 2.3 is false in general, even if $M \in \mathscr{M}$ and is additionally assumed to be a Lipschitzian submanifold of $\mathbb{R}^n$. In $\mathbb{R}^2$, consider the square $M = \{(x,y) \in \mathbb{R}^2 \mid \sup\{|x|,|y|\} = 1\}$ and note that $\chi(M) = 0$ and $N_M(1,1) = \mathbb{R}^2$.

3. Proof of the existence result (Theorem 2.2)

The proof of Theorem 2.2 goes as follows. First, in Section 3.1, we reduce the proof to the case of a connected set M. Then, in Section 3.2, approximating both the correspondence F and the set M (or a connected component of M), we deduce the existence of generalized equilibria from known results on smooth sets (Poincaré–Hopf theorem) by a limit argument, noticing that $\chi(M_k) = \chi(M) \neq 0$ for every k (from the retraction condition (ret)). Finally, in Section 3.3, we deduce the existence of equilibria (Part (b)).

3.1. Reducing to the connected case

The following result allows us to reduce the proof of Theorem 2.2 to the case where M is connected: in this case, Assertion (χ) reduces to the assertion $\chi(M) \neq 0$.

Proposition 3.1. *Let $M \in \mathscr{A}$, then M has a finite number of connected components $(M^i)_{i\in I}$, and, for every $i \in I$, there is an open subset U^i of $\mathbb{R}^n$ such that $U^i \cap M = M^i$. Furthermore, for all $i \in I$, M^i is nonempty, compact, belongs to $\mathscr{A}$, one has*

$$\tilde{N}_{M^i}(x) = \tilde{N}_M(x), \quad N_{M^i}(x) = N_M(x)$$
and
$$T_{M^i}(x) = T_M(x) \text{ for every } x \in M^i \tag{2}$$

and $\chi(M) = \sum_{i\in I} \chi(M^i)$.

Proof. Let (M_k) be a smooth normal approximation of M, in the sense of Definition 2.2. We consider the set M_0, the first term of the sequence (M_k) and we let $(M_0^i)_{i\in I_0}$ be the connected components of the set M_0. Since M_0 is smooth and compact, the set I_0 is finite, and for all $i \in I_0$, there is an open subset U^i of $\mathbb{R}^n$ such that $M_0^i = U^i \cap M_0$ is smooth and compact (this is a well-known fact from differential topology). Since $M \subset M_0 \subset \bigcup_{i\in I_0} U^i$, then $M = \bigcup_{i\in I_0} U^i \cap M$. If $i \neq j$, then $(U^i \cap M) \cap (U^j \cap M) \subset (U^i \cap M_0) \cap (U^j \cap M_0) = \emptyset$.

Let $i \in I_0$, we now prove that $U_i \cap M \in \mathscr{A}$ and that $U_i \cap M$ is nonempty and connected. Condition (2) is then clear since it is a local property. For every k, the set $U^i \cap M_k$ is clearly smooth. It is also compact, since $U^i \cap M_k = U^i \cap M_0 \cap M_k = M_0^i \cap M_k$, which is compact. Indeed, $U^i \cap M_k \subset U^i \cap M_0 \cap M_k \subset U^i \cap M_k$.

The sequence $(U^i \cap M_k)_{k\in\mathbb{N}}$ clearly satisfies Assertions (ii) and (cn). It also satisfies Assertion (ret) by considering, for every k, the map $H_k^i = H_k|_{(M_k \cap U_i)\times[0,1]}$. This shows in particular that $H_0^i(M_0, 1) = U^i \cap M$ is nonempty and connected. Hence the sets $(U^i \cap M)_{i\in I_0}$ are the connected components of M and, from above, each component $U^i \cap M$ belongs to $\mathscr{A}$.

Since the Euler characteristic is invariant by a deformation retraction, then $\chi(M) = \chi(M_0)$ and, from above, $\chi(M_0^i) = \chi(U^i \cap M)$ for every $i \in I_0$. Furthermore, one has $\chi(M_0) = \sum_{i\in I_0} \chi(M_0^i) = \sum_{i\in I_0} \chi(U^i \cap M)$, hence $\chi(M) = \sum_{i\in I_0} \chi(U^i \cap M)$. □

3.2. *Proof of Part* (a)

The implication $[(GE; \tilde{N}_M) \Rightarrow (GE; N_M)]$ is immediate since $\tilde{N}_M(x) \subset N_M(x)$ for every $x \in M$.

Proof of $[(\chi) \Rightarrow (GE; \tilde{N}_{\mathrm{M}})]$**.** Without any loss of generality, from Proposition 3.1, we may assume that M is connected. Let F be an u.s.c. correspondence from M to $\mathbb{R}^n$, with nonempty compact convex values, and let (M_k) be a smooth normal approximation of M. We need first to extend the correspondence F to the whole space and then to approximate it by a continuous single-valued map. Indeed, from Cellina [5], there is a bounded u.s.c. correspondence $\hat{F}$, with nonempty, compact, convex values, defined on $\mathbb{R}^n$ with values in $\mathbb{R}^n$, such that $\hat{F}(x) = F(x)$ for every $x \in M$. Furthermore, from Cellina [5], for every $k \in \mathbb{N}$, there is a continuous map $\hat{f}_k : R^n \to \mathbb{R}^n$ such that $\mathrm{G}(\hat{f}_k) \subset B(\mathrm{G}(\hat{F}), 1/k)$.

Claim 3.1. *For every k, there is $x_k \in M_k$ such that $\hat{f}_k(x_k) \in N_{M_k}(x_k)$.*

Proof. Since the set M_k is smooth, for every $x \in \mathrm{bd}\, M_k$, the set $N_{M_k}(x) \cap S$ is reduced to a singleton, denoted $g_{M_k}(x)$, and the map $g_{M_k} : \mathrm{bd}\, M_k \to S$ is continuous. Again, we extend g_{M_k} to the whole space as follows. From [19], there is a continuous map $\hat{g}_{M_k}$ from $\mathbb{R}^n$ to $\bar{B}(0,1)$ such that $\hat{g}_{M_k}(x) = g_{M_k}(x)$ for every $x \in \mathrm{bd}\, M_k$. For every integer $v \geq 1$, let $\alpha_k^v : M_k \to [0,1]$ be a continuous map such that $\alpha_k^v(x) = 1$ for every $x \in \mathrm{bd}\, M_k$ and $\alpha_k^v(x) = 0$ if $d_{\mathrm{bd}\, M_k}(x) \geq 1/v$. We now define an inward tangent field $\varphi_k^v : M_k \to \mathbb{R}^n$ on M_k as follows:

$$\varphi_k^v(x) = \hat{f}_k(x) - \lambda_k^v \alpha_k^v(x) \hat{g}_{M_k}(x), \tag{3}$$

where $\lambda_k^v = 1/v + \max\{0, \max\{(\hat{f}_k(x) | g_{M_k}(x)) | x \in \mathrm{bd} M_k\}\}$.

Since the Euler characteristic is invariant by a deformation retraction, from Assertion (ret), $\chi(M_k) = \chi(M)$ for all k. Hence $\chi(M_k) = \chi(M) \neq 0$ from Assertion (χ) and our assumption that M is connected. Noting that φ_k^v points inward on the boundary of M_k, i.e., $(\varphi_k^v(x) | g_{M_k}(x)) < 0$ for every $x \in \mathrm{bd}\, M_k$, from Poincaré–Hopf theorem (see [22]), there is $x_k^v \in M_k$ such that $\varphi_k^v(x_k^v) = 0$. Since the set M_k, the map α_k^v, and λ_k^v are bounded, without any loss of generality, the sequence $(x_k^v, \alpha_k^v(x_k^v), \lambda_k^v)_{v \in \mathbb{N}}$ converges to some $(x_k, \alpha_k, \lambda_k) \in M_k \times \bar{B}(0,1) \times \mathbb{R}_+$ when $v \to \infty$. Taking the limit in (3) when $v \to \infty$, one gets $\hat{f}_k(x_k) = \lambda_k \alpha_k \hat{g}_{M_k}(x_k)$. If $x_k \in \mathrm{int}\, M_k$, then for v large enough, $d_{\mathrm{bd}\, M_k}(x_k^v) \geq 1/v$, $\alpha_k^v(x_k^v) = 0$, $\alpha_k = 0$ and $\hat{f}_k(x_k) = 0 \in N_{M_k}(x_k)$. If $x_k \in \mathrm{bd}\, M_k$, then $\hat{f}_k(x_k) = \lambda_k \alpha_k g_{M_k}(x_k) \in N_{M_k}(x_k)$. □

We now come back to the proof of $[(\chi) \Rightarrow (GE; \tilde{N}_M)]$. Since $B(M,1)$ is bounded and since $(\hat{f}_k)$ is uniformly bounded, without any loss of generality, the sequence $(x_k, \hat{f}_k(x_k))$ converges to some $(x^*, p^*) \in \mathbb{R}^n \times \mathbb{R}^n$. From Claim 3.1, for every k, $x_k \in M_k$ and $\hat{f}_k(x_k) \in N_{M_k}(x_k)$, and since $\limsup \mathrm{G}(N_{M_k}) \subset \mathrm{G}(\tilde{N}_M)$, we get $x^* \in M$ and $p^* \in \tilde{N}_M(x^*)$. Recalling that, for every k, $\mathrm{G}(\hat{f}_k) \subset B(\mathrm{G}(\hat{F}), 1/k)$, we deduce that $(x_k, \hat{f}_k(x_k)) \in B(\mathrm{G}(\hat{F}), 1/k)$. Taking the limit when $k \to \infty$, using the fact that $\mathrm{G}(\hat{F})$ is closed (since $\hat{F}$ is u.s.c.), we get that $(x^*, p^*) \in \mathrm{G}(\hat{F})$, i.e., $p^* \in \hat{F}(x^*)$. Since $x^* \in M$, $p^* \in \hat{F}(x^*) = F(x^*)$. Thus $0 \in F(x^*) - \tilde{N}_M(x^*)$. □

3.3. *Proof of Part* (b)

The equivalence $[(E; \tilde{N}_M) \Leftrightarrow (E; N_M)]$ is always true. Indeed, for every $x \in M$, $\mathrm{cl}(\mathrm{co}\, \tilde{N}_M(x)) = N_M(x)$ (Proposition 2.1), which implies that $\tilde{N}_M(x)^\circ = N_M(x)^\circ$.

In Part (a), we have proved the implication $[(\chi) \Rightarrow (GE; \tilde{N}_M) \Rightarrow (GE; N_M)]$. The proof of Part (b) is a consequence of the following proposition, taking $N = \tilde{N}_M$ if $\tilde{N}_M$ has a closed graph, or $N = N_M$ if N_M has a closed graph.

Proposition 3.2. *Let $M \subset \mathbb{R}^n$ be a compact set, let N be a correspondence from M to $\mathbb{R}^n$ with a closed graph, such that $N(x)$ is a cone for every $x \in M$. Then:*

$$(GE; N) \Rightarrow (E; N).$$

Proof. Let F be an u.s.c. correspondence from M to $\mathbb{R}^n$, with convex compact values, such that $F(x) \cap N(x)^\circ \neq \emptyset$ for every $x \in M$. We let, for $x \in M$ and $k \in \mathbb{N}$:

$$F_k(x) = \mathrm{coB}(F(B(x, 1/k) \cap M), 1/k),$$

$$T_k(x) = \{y \in \mathbb{R}^n \mid \forall p \in N(x) \cap S, (y \mid p) < 1/k\},$$

$$\Phi_k(x) = F_k(x) \cap T_k(x)$$

and we claim that Φ_k admits a continuous selection. Formally:

Claim 3.2. *For every k, there exists a continuous map $\varphi_k : M \to \mathbb{R}^n$ such that, for all $x \in M$, $\varphi_k(x) \in F_k(x) \cap T_k(x)$.*

Admitting the claim (the proof of which is given below), we then proceed as follows. Let φ_k be given by the above claim, then from $(GE;N)$ there is $x_k \in M$ such that $0 \in \varphi_k(x_k) - N(x_k)$. We show that, for every k, $\|\varphi_k(x_k)\| < 1/k$. It is immediate if $\varphi_k(x_k) = 0$. If $\varphi_k(x_k) \neq 0$, recalling that $\varphi_k(x_k) \in T_k(x_k)$, we get $\|\varphi_k(x_k)\| = (\varphi_k(x_k) \mid \varphi_k(x_k)/\|\varphi_k(x_k)\|) < 1/k$. Without any loss of generality, we may assume that the sequence (x_k) converges to some element $x^* \in M$. Since, for every k, $\|\varphi_k(x_k)\| \leq 1/k$, then the sequence $(\varphi_k(x_k))$ converges to 0. Since $\varphi_k(x_k) \in F_k(x_k)$, from Carathéodory's theorem, there are $n+1$ elements $(x_k^i, y_k^i, \lambda_k^i)$ in $M \times \mathbb{R}^n \times [0,1]$ such that, for every $i \in \{1, \ldots, n+1\}$, $x_k^i \in B(x_k, 1/k) \cap M$, $y_k^i \in B(F(x_k^i), 1/k)$, and such that

$$1 = \sum_{i=1}^{n+1} \lambda_k^i,$$

$$\varphi_k(x_k) = \sum_{i=1}^{n+1} \lambda_k^i y_k^i. \tag{4}$$

Then, for every $i \in \{1, \ldots, n+1\}$ the sequence $(x_k^i)_{k \in \mathbb{N}}$ converges to x^*. Without any loss of generality, we may assume that the sequence $(y_k^1, \ldots, y_k^{n+1}, \lambda_k^1, \ldots, \lambda_k^{n+1})_{k \in \mathbb{N}}$ converges to some element $(y^1, \ldots, y^{n+1}, \lambda^1, \ldots, \lambda^{n+1})$ in the bounded set $B(F(M), 1)^{n+1} \times [0,1]^{n+1}$. Since the correspondence F is u.s.c., for every $i \in \{1, \ldots, n+1\}$, $y^i \in F(x^*)$. Taking the limit in (4), and since F has convex values, $1 = \sum_{i=1}^{n+1} \lambda^i$, $0 = \sum_{i=1}^{n+1} \lambda^i y^i \in F(x^*)$. Hence the x^* is an equilibrium of F, hence $(E;N)$ holds. □

Proof of the Claim. The correspondence Φ_k has clearly convex values. Since, for all x, $F(x) \cap N(x)^\circ \neq \emptyset$ and since $N(x)^\circ \subset T_k(x)$, Φ_k has nonempty values. In view of Michael's selection theorem (see [20]),[7] the proof of Claim 3.2 is completed if we show that the correspondence Φ_k is lower semicontinuous. We show in fact the stronger condition that the set $\{x \in M \mid y \in \Phi_k(x)\}$ is open in M (for its relative topology), for every $y \in \mathbb{R}^n$. Indeed, let $(x, y) \in M \times \mathbb{R}^n$ such that $y \in \Phi_k(x) = F_k(x) \cap T_k(x)$.

[7] In fact, we use the following simple case of Michael's selection theorem, which can be proved directly. If Φ is a correspondence defined on a nonempty compact subset M of $\mathbb{R}^n$, with nonempty convex values in $\mathbb{R}^m$, and such that, for all $y \in \mathbb{R}^m$, the set $\{x \in \mathbb{R}^n \mid y \in \Phi(x)\}$ is open in M (for its relative topology), then Φ admits a continuous selection.

Since $y \in F_k(x)$, from Carathéodory's theorem, there are $n+1$ elements (x^i, y^i, λ^i) in $M \times \mathbb{R}^n \times [0,1]$ such that, for all $i \in \{1,\dots,n+1\}$, $x^i \in B(x, 1/k)$, $y^i \in B(F(x^i), 1/k)$, $\sum_{i=1}^{n+1} \lambda^i = 1$, and such that: $y = \sum_{i=1}^{n+1} \lambda^i y^i$. We let $r = 1/k - \max\{\|x - x^i\| | i \in \{1,\dots,n+1\}\}$, then for all $x' \in B(x,r) \cap M$ (noticing that $x^i \in B(x', 1/k)$), one gets $y \in F_k(x')$. Since $y \in T_k(x)$, $N(x) \cap S \subset \{p \in \mathbb{R}^n | (y|p) < 1/k\}$. Since the correspondence $N \cap S$ is u.s.c., by eventually taking a smaller $r > 0$, we may assume that, for every $x' \in B(x;r) \cap M$, $N(x') \cap S \subset \{p \in \mathbb{R}^n \mid (y|p) < 1/k\}$, hence $y \in T_k(x')$. Consequently, for every $x' \in B(x;r) \cap M$, $y \in \Phi_k(x') = F_k(x') \cap T_k(x')$. $\square$

4. Proof of the approximation result (Theorem 2.1)

Let $M \in \mathscr{M}$. We first show that the correspondence $\tilde{N}_M$ has a closed graph. Indeed, let (x_k, y_k) be a sequence in $M \times \mathbb{R}^n$, converging to some $(x, y) \in M \times \mathbb{R}^n$ and such that, for every k, $y_k \in \tilde{N}_M(x_k)$. Hence there is $\lambda_k \geq 0$ and $z_k \in \partial_+ d_M(x_k) \subset \bar{B}(0,1)$ such that $y_k = \lambda_k z_k$. Without any loss of generality, we may assume that (z_k) converges to some $z \in \mathbb{R}^n$. Since the correspondence $\partial_+ d_M$ is u.s.c. (Proposition 2.1), then $z \in \partial_+ d_M(x)$, which does not contain 0 since $M \in \mathscr{M}$. Then (λ_k) converges to $\lambda = \|y\|/\|z\|$, hence $y = \lambda z$ belongs to $\lambda \partial_+ d_M(x)$, hence also to $\tilde{N}_M(x)$. $\square$

We now show the existence of a smooth normal approximation of M (in the sense of Definition 2.2). For this, we need the following representation result from [15] (see also [12]). The proof given in [15], consists in "smoothing" the distance function d_M by using a convolution-type argument and keeping its "good" properties related to the subdifferential.

Theorem 4.1. *Let M be a closed subset of $\mathbb{R}^n$. Then there is a Lipschitzian map $f_M : \mathbb{R}^n \to \mathbb{R}_+$ such that*

(a) $M = \{x \in \mathbb{R}^n \mid f_M(x) = 0\}$;
(b) f_M *is* C^∞ *on* $\mathbb{R}^n \setminus M$;
(c) $\partial f_M(x) \subset \partial d_M(B(x, d_M(x))) + B(0, d_M(x))$, *for every* $x \in \mathbb{R}^n$;
(d) $\forall x \in M$, $\partial_+ f_M(x) \subset \partial_+ d_M(x)$, *hence* $\bigcup_{\lambda \geq 0} \lambda \partial_+ f_M(x) \subset \tilde{N}_M(x)$;
(e) $f_M^{-1}((0, 1/2]) \subset B(\mathrm{bd}\, M, 1)$.

4.1. Proof of Theorem 2.1

Let f_M be a map given by Theorem 4.1. If we additionally assume that M is proximally nondegenerate and compact, we now show that there is $\beta \in (0, \frac{1}{2}]$ such that M satisfies the following nondegeneracy assumption:

(f) $\|\nabla f_M(x)\| > \beta$ for every $x \in f_M^{-1}((0, \beta])$.

Indeed, from the definition of proximally nondegenerate sets, since bd M is compact, and from Assertion (e) of Theorem 4.1, there are $\alpha > 0$ and $\beta_0 > 0$ such that:

$$\forall x \in B(M,\alpha) \setminus M, \forall p \in \partial d_M(x), \|p\| \geq \alpha;$$
$$f_M^{-1}((0, \beta_0]) \subset B(M, \alpha/2) \setminus M.$$

Let $\beta = \min\{\alpha/2, \beta_0\}$ and consider $x \in f_M^{-1}((0,\beta])$. Then $x \notin M$ and f_M is differentiable at x. From Assertion (c) of Theorem 4.1:

$$\nabla f_M(x) \in \partial d_M(B(x, d_M(x))) + B(0, \alpha/2),$$

hence from above $\|\nabla f_M(x)\| \geq \alpha/2 \geq \beta$. This ends the proof of Assertion (f). □

We now define the sequence (M_k) as follows:

$$M_k = \{x \in \mathbb{R}^n \mid f_M(x) \leq 1/k\}$$

and we show that, for $k_0 > 1/\alpha$, the sequence $(M_k)_{k \geq k_0}$ satisfies the conclusions of Theorem 2.1.

Proof of (i). For every $k \geq k_0$, M_k is compact from Assertion (e) of Theorem 4.1 and the continuity of f_M. It is smooth from Assertion (c) of Theorem 4.1. □

Proof of (ii). From the definition of the sets M_k, it is immediate that the sequence (M_k) is decreasing and that $M = \bigcap_k M_k$. From Asseriton (e) of Theorem 4.1, $M_k \subset B(M, 1)$. □

Proof of (cn). Let $(x, p) \in \limsup_{k\to\infty} \mathrm{G}(N_{M_k})$. Without any loss of generality, we may assume that there is a sequence (x_k, p_k) of elements of $\mathbb{R}^n \times \mathbb{R}^n$ converging to (x, p), such that, for every k, $x_k \in M_k$ and $p_k \in N_{M_k}(x_k)$. Since $x_k \in M_k$, $f_M(x_k) \leq 1/k$, hence at the limit we get $f_M(x) \leq 0$. Recalling that $f_M(x) \geq 0$, we thus deduce that $x \in M$. From [6], since $\nabla f_M(x) \neq 0$ whenever $f_M(x) = 1/k$, we recall that

$$N_{M_k}(x) = \bigcup_{\lambda \geq 0} \lambda \nabla f_M(x) \quad \text{if } f_M(x) = 1/k,$$

$$N_{M_k}(x) = \{0\} \quad \text{if } f_M(x) < 1/k.$$

We now distinguish the two cases $p = 0$ and $p \neq 0$. If $p = 0$, then clearly $p \in \tilde{N}_M(x)$. If $p \neq 0$, for k large enough, $p_k \neq 0$, hence from above, there is $\lambda_k \geq 0$ such that $p_k = \lambda_k \nabla f_M(x_k)$. Since f_M is Lipschitzian, the sequence $(\nabla f_M(x_k))$ is bounded; without any loss of generality, we may assume that $\nabla f_M(x_k)$ converges to some $v \in \mathbb{R}^n$. From the definition of the set $\partial_+ f(x)$, we get that $v \in \partial_+ f_M(x)$. But from the above assertion (f), for k large enough, $\|\nabla f_M(x_k)\| \geq \alpha$, which at the limit implies that $\|v\| \neq 0$. Hence the sequence (λ_k) converges to $\lambda = \|p\|/\|v\|$, $p = \lambda v$, and $p \in \lambda \partial_+ f_M(x) \subset \tilde{N}_M(x)$. □

Proof of (ret). Assertion (ret) is a direct consequence of the following deformation lemma.

Lemma 4.1. *Let a, b be real numbers, let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitzian function such that $f^{-1}([a,b])$ is compact, and assume that f is C^2 on an open neighborhood of $f^{-1}((a,b])$, and satisfies the nondegeneracy assumption*:

$$\exists \alpha > 0, \quad \forall x \in f^{-1}((a,b]), \ \|\nabla f(x)\| \geq \alpha.$$

Then $M_\alpha = f^{-1}((-\infty, a])$ is a strong deformation retract of $M_b = f^{-1}((-\infty, b])$, i.e., there is a continuous map $H : [0,1]\times M_b \to M_b$ such that, for every $x \in M_b$, $H(0,x) = x$, and $H(1,x) \in M_a$, and, for every $x \in M_a$, for every $t \in [0,1]$, $H(t,x)=x$. Furthermore, we may assume that there is $k > 0$ such that $\|H(1,x)-x\| \le k d_{M_a}(x)$, for every $x \in M_b$.

The above lemma generalizes parts of the "noncritical neck principle" as discussed by Milnor [21, Theorem 3.1] and Schwartz [26, Theorems 4.7.1 and 4.7.2], who assume that f is C^2 on $f^{-1}([a,b])$ and that $\nabla f(x) \neq 0$ for every $x \in f^{-1}([a,b])$, but allow the space to be an (infinite-dimensional) complete Riemannian manifold. See also Bonnisseau–Cornet [4] in the Lipschitzian case. The proof of Lemma 4.1 is given in the next section.

4.2. Proof of the deformation lemma

As in the smooth case, we shall define the homotopy map H using the flow of the gradient field of f defined on $f^{-1}((a,b])$. Precisely, we consider the following differential equation, for $x \in M_b \setminus M_a$:

$$\dot{x}(t) = -\nabla f(x(t))/\|\nabla f(x(t))\|^2, \quad x(0) = x$$

and we denote $\varphi(.,x)$ its maximal solution, which is defined on an interval $I(x)$, with values in $M_b \setminus M_a$. We prepare the proof of the deformation lemma with a claim.

Claim 4.1. *Let $x \in M_b \setminus M_a$. Then*:
(i) $\sup I(x) = f(x) - a$;
(ii) $\lim_{t\to f(x)-a} \varphi(t,x)$ *exists and belongs to* $f^{-1}(\{a\})$.

Proof of (i). Note that, for every $t \in I(x)$,

$$\frac{\mathrm{d}(f_M \circ \varphi(.,x))}{\mathrm{d}t}(t) = (\nabla f(\varphi(t,x))|(\partial\varphi/\partial t)(t,x)) = -\frac{\|\nabla f(\varphi(t,x))\|^2}{\|\nabla f(\varphi(t,x))\|^2} = -1.$$

Thus, for every $t \in I(x)$, $a < f(\varphi(t,x)) = f(\varphi(0,x)) - t = f(x) - t$. Hence $\sup I(x) \le f(x) - a$. One deduces the equality, since $\varphi(.,x)$ is a maximal solution. □

Proof of (ii). For every t and t' in $[0, f(x) - a)$,

$$\varphi(t,x) - \varphi(t',x) = \int_{t'}^{t} \frac{\partial\varphi}{\partial t}(u,x)\,\mathrm{d}u = \int_{t'}^{t} -\frac{\nabla f(\varphi(u,x))}{\|\nabla f(\varphi(u,x))\|^2}\,\mathrm{d}u,$$

which implies, together with the nondegenracy assumption in Lemma 4.1

$$\|\varphi(t,x) - \varphi(t',x)\| \le |t - t'|/\alpha, \tag{5}$$

hence $\varphi(.,x)$ is Lipschitzian on $[0, f(x) - a)$ and can be continuously extended on $[0, f(x) - a]$, thus $\lim_{t\to f(x)-a} \varphi(t,x)$ exists. Recalling from above that

$$f(\varphi(t,x)) = f(x) - t,$$

we deduce that $f(\lim_{t\to f(x)-a} \varphi(t,x)) = a$. □

We now come back to the proof of Lemma 4.1. We define the map $h : \mathbb{R}_+ \times M_b \to M_b$ as follows, for $(t,x) \in \mathbb{R}_+ \times M_b$:

$$h(t,x) = \varphi(t,x) \qquad \text{if } f(x) \in (a,b] \quad \text{and} \quad t < f(x) - a,$$

$$h(t,x) = \lim_{\tau \to f(x)-a} \varphi(\tau,x) \qquad \text{if } f(x) \in (a,b] \quad \text{and} \quad t \geq f(x) - a,$$

$$h(t,x) = x \qquad \text{if } f(x) \leq a.$$

We now show that M_a is a strong deformation retract of M_b via the homotopy $H : [0,1] \times M_b \to M_b$ defined by

$$H(t,x) = h((b-a)t,x) \quad \text{for every } (t,x) \in [0,1] \times M_b.$$

Indeed, let $x \in M_b$, then $H(0,x) = h(0,x) = x$; if $b \geq f(x) > a$, then $H(1,x) = h(b - a,x) = h(f(x) - a,x) \in f^{-1}(\{a\})$; if $f(x) \leq a$, then $H(t,x) = h((b-a)t,x) = x$. The fact that the map H is continuous is a consequence of the following claim.

Claim 4.2. *The map $h : \mathbb{R}_+ \times M_b \to M_b$ is continuous.*

Proof. Let $(t,x) \in \mathbb{R}_+ \times M_b$, we show that h is continuous at (t,x). Let us first notice that for every $(t',x') \in \mathbb{R}_+ \times M_b$,

$$\|h(t,x) - h(t',x')\| \leq \|h(t,x) - h(t,x')\| + \|h(t,x') - h(t',x')\|. \tag{6}$$

We now show that

$$\forall (t',x') \in \mathbb{R}_+ \times M_b, \quad \|h(t,x') - h(t',x')\| \leq |t - t'|/\alpha. \tag{7}$$

Indeed, if $f(x') \leq a$, the inequality is clear since $h(t,x')=h(t',x')=x'$. We now consider the case $f(x') > a$. If $\max\{t,t'\} < f(x') - a$, the above inequality is a consequence of (5). If $t < f(x') - a$ and $t' \geq f(x') - a$, from the definition of f, $\|h(t,x') - h(t',x')\| = \|\varphi(t,x') - \lim_{\tau \to f(x)-a} \varphi(\tau,x)\|$. But from (5), $\|\varphi(t,x') - \varphi(\tau,x')\| \leq |t - \tau|/\alpha$, hence $\|\varphi(t,x') - \lim_{\tau \to f(x')-a} \varphi(\tau,x')\| \leq (f(x') - a - t)/\alpha \leq (t' - t)/\alpha$. If $\min\{t,t'\} \geq f(x') - a$, the result is clear since $h(t,x') = h(t',x')$ from the definition.

In view of (6), the proof will be complete if we show that (for the given t)

$$\lim_{x' \to x} h(t,x') = h(t,x). \tag{8}$$

To show this, we distinguish the following three cases:

Case 1: $f(x) \leq a$. Then $h(t,x)=x$. Let $x' \in M_b$. If $f(x') \leq a$, then $\|h(t,x) - h(t,x')\| = \|x - x'\|$. If $f(x') > a$, then $\|h(t,x) - h(t,x')\| \leq \|x - x'\| + \|x' - h(t,x')\| \leq \|x - x'\| + \|x' - \lim_{\tau \to f(x')-a} \varphi(\tau,x')\| \leq \|x - x'\| + (f(x') - a)/\alpha \leq \|x - x'\| + (f(x') - f(x))/\alpha$ from Claim 4.1. Thus, in both cases $\|h(t,x) - h(t,x')\| \leq (1 + l/\alpha)\|x - x'\|$, where L is the lipschitz constant of f.

Case 2: $f(x) > a$ *and* $t < f(x) - a$. Then $h = \varphi$ on a neighborhood of (t,x), and the continuity of φ at (t,x) is a well-known result (see, for example [19]).

Case 3: $f(x) > a$ *and* $t \geq f(x) - a$. Then $h(t,x) = h(f,x) - a,x)$ from the definition. Let $\varepsilon > 0$ be given. We let $\tau = \max\{f(x) - a - \alpha\varepsilon, 0\}$. We choose a neighborhood U of x such that, for every $x' \in U$, one has $|f(x)-f(x')| < \alpha\varepsilon$, $f(x') > a$, and $\|\varphi(\tau,x)-\varphi(\tau,x')\| < \varepsilon$. Then $\tau < f(x)-a$ and $\tau < f(x')-a$, hence $h(\tau,x)=\varphi(\tau,x)$ and $h(\tau,x') = \varphi(\tau,x')$. Noting that $\|h(t,x) - h(t,x')\|$ is equal to

$$\|h(f(x) - a,x) - h(\tau,x) + h(\tau,x) - h(\tau,x') + h(\tau,x') - h(\min\{t, f(x') - a\},x')\|$$

and in view of (7), we deduce

$$\begin{aligned}\|h(t,x) - h(t,x')(\|L(t,x) - a(t,x)\|) &\le (f(x) - a - \tau)/\alpha + \|\varphi(\tau,x) - \varphi(\tau,x')\| \\ &\quad +(\min\{t, f(x') - a\} - \tau)/\alpha \\ &\le \varepsilon + \|\varphi(\tau,x) - \varphi(\tau,x')\| + 2\varepsilon \le 4\varepsilon. \qquad \square\end{aligned}$$

Finally, the proof of Lemma 4.1 is completed with the following claim.

Claim 4.3. *There is $k > 0$ such that $\|h(b - a,x) - x\| \le kd_{M_a}(x)$, for every $x \in M_b$.*

Proof. Let $x \in M_b$. If $f(x) \le a$, then $h(b-a,x)=x$ and the result is clear. If $f(x) > a$, then $h(b-a,x)=h(f(x)-a,x)$ from the definition of h, hence from (7), $\|h(b-a,x)-x\|=\|h(f(x)-a,x)-h(0,x)\| \le (f(x)-a)/\alpha$. Let $y \in M_a$ such that $\|x-y\|=d_{M_a}(x)$, then, if l is the Lipschitz constant of f, we get $f(x)-a=f(x)-f(y) \le l\|x-y\|=ld_{M_a}(x)$ and $\|h(b - a,x) - x\| \le (l/\alpha)d_{M_a}(x)$. $\square$

5. Concluding remarks

5.1. Other notions of normal cones

Up to now, we have considered the two cases $N = N_M$ (Clarke's normal cone) and $N = \tilde{N}_M(x)$. The following remarks discuss the existence problem of equilibria and generalized equilibria, when one considers other notions of normal cones such as $N = \hat{N}_M$, the limiting normal cone and $N = N_M^{\mathrm{B}}$, Bouligand normal cone. [8]

5.1.1. The case $N = \hat{N}_M$, the limiting normal cone

Remark 1 (*Generalized equilibria*)**.** The implication $[(\chi) \Rightarrow (GE;\hat{N}_M)]$ may not be true, even if the set M is compact and epi-Lipschitzian. In $\mathbb{R}^3$, consider the counter-example in [7], $\chi(M) = 1$ (for example, M is homeomorphic to a convex set).

Remark 2 (*Equilibria*)**.** However, the implication $[(\chi) \Rightarrow (E;\hat{N}_M)]$ holds under the assumption of Theorem 2.2, Part (b), since $\mathrm{cl}(\mathrm{co}\hat{N}_M(x)) = N_M(x)$ (Proposition 2.1), which implies the equivalence $[(E;\hat{N}_M) \Leftrightarrow (E;N_M)]$.

[8] We recall that Bouligand normal cone to M at x is defined by $N_M^{\mathrm{B}}(x) = T_M^{\mathrm{B}}(x)^{\circ}$, where $T_M^{\mathrm{B}}(x) = \{v \in R^n \mid \exists(\lambda_k)_{k\in\mathbb{N}},\ \lambda_k > 0,\ \exists(y_k)_{k\in N}, y_k \in M,\ y_k \to x,\ v = \lim_{k\to\infty} \lambda_k(y_k - x)\}$.

5.1.2. The case $N = N_M^{\mathrm{B}}$, Bouligand normal cone

Remark 3 (*Generalized equilibria*)**.** The implication $[(\chi) \Rightarrow (GE; N_M^{\mathrm{B}})]$ may not be true, even if the set M is compact and epi-Lipschitzian. Consider the counterexample in [7] and note that $N_M^{\mathrm{B}}(0) = \{0\}$.

Remark 4 (*Equilibria in the multi – valued case*)**.** The implication $[(\chi) \Rightarrow (E; N_M^{\mathrm{B}})]$ may not be true, even if the set M is compact and epi-Lipschitzian. Consider the set M in the counterexample in [7]. Then $\chi(M) = 1$ since M is homeomorphic to a convex set. Hence from Theorem 2.2. Assertion $(E; N_M)$ holds true. But Assertion $(E; N_M^{\mathrm{B}})$ holds false. In this case, the implications $[(\chi) \Rightarrow (E; N_M^{\mathrm{B}})]$ and $[(E; N_M) \Rightarrow (E; N_M^{\mathrm{B}})]$ holds false.

Remark 5 (*Equilibria in the single – valued case*)**.** The implication $[(\chi) \Rightarrow (E_{sv}; N_M^{\mathrm{B}})]$ holds under the assumptions of Theorem 2.2, Part (*a*). Indeed, let M be a closed subset of $\mathbb{R}^n$ and let $f : M \to \mathbb{R}^n$ be a continuous map; then from [10], the two following assertions are equivalent:

(i) $f(x) \in T_M^{\mathrm{B}}(x)$ for every $x \in M$;
(ii) $f(x) \in T_M(x)$ for every $x \in M$.

Hence, the equivalence $[(E_{sv}; N_M) \Rightarrow (E_{sv}; N_M^{\mathrm{B}})]$ holds.

5.2. Other classes of sets

We give further relations between the class of proximally nondegenerate sets, introduced in this paper, and other related sets considered in the literature for similar problems, such as $\mathscr{L}$-retract, Lipschitzian submanifolds, proximally regular sets and proximally smooth sets.

5.2.1. $\mathscr{L}$-retracts

We recall that a closed set $M \subset \mathbb{R}^n$ is said to be an $\mathscr{L}$-retract [2], if there are an open neighborhood U of M, a continuous map $r : U \to M$, and $k \geq 0$, such that,

$$\|x - r(x)\| \leq k d_M(x) \quad \text{for every } x \in U.$$

Note that this implies that r is a retraction, i.e., for every $x \in M$, $r(x) = x$. This definition was introduced by [2] in the more general setting of a metric space.

Proposition 5.1. *If $M \in \mathscr{M}$ is compact, then M is a $\mathscr{L}$-retract.*

Proof. Let $M \in \mathscr{M}$, let M_1 and H be defined as in Theorem 2.1 and assume that H additionally satisfies that, for some $k > 0$, $\|H(1,x) - x\| \leq k d_M(x)$ for every $x \in M_1$ (as shown in Lemma 4.1). Then we let $r(x) = H(1,x)$ for every $x \in \operatorname{int} M_1$ and $\|x - r(x)\| \leq k d_M(x)$. □

Remark 6. The class $\mathscr{A}$ of approximable sets, introduced in Definition 2.2, is not contained in the class of $\mathscr{L}$-retracts. In $\mathbb{R}^2$, consider

$$M = M_1 \cup M_2 \cup [-1, 1] \times \{0\}, \text{ where,}$$

$$M_1 = \{(x, y) \in \mathbb{R}^2 \mid x \leq 1 \quad \text{and} \quad (x-1)^2 + y^2 = 1\},$$
$$M_2 = \{(x, y) \in \mathbb{R}^2 \mid x \geq -1 \quad \text{and} \quad (x+1)^2 + y^2 = 1\}.$$

5.2.2. Proximally smooth and proximally regular sets

We recall that a closed set $M \subset \mathbb{R}^n$ is said to be proximally regular (see [23,24]) [resp. proximally smooth (see [8]) also called sets of positive real (Federer [18])], if, for every $x \in M$, there is $\alpha > 0$ such that d_M is differentiable on $B(x, \alpha) \setminus M$ [resp. $B(M, \alpha) \setminus M$].

Proposition 5.2. *If M is proximally regular, then $M \in \mathscr{M}$.*

Proof. Let M be proximally regular and let $\bar{x} \in M$. Then there is $\alpha > 0$ such that d_M is differentiable on $B(\bar{x}, \alpha) \setminus M$. Let $x \in B(\bar{x}, \alpha) \setminus M$, from [6, Proposition 2.5.4], there is a unique $p(x) \in M$ such that $\|x - p(x)\| = d_M(x)$ and $\nabla d_M(x) = (x - p(x))/\|x - p(x)\|$. Noting that the single-valued map $x \mapsto p(x)$ is continuous, one deduces that ∇d_M is continuous on $B(\bar{x}, \alpha) \setminus M$, hence $\partial d_M(x) = \{\nabla d_M(x)\} \subset S$ for every $x \in B(\bar{x}, \alpha) \setminus M$. Consequently, $0 \notin \partial_+ d_M(\bar{x}) \subset S$. $\square$

5.2.3. Lipschitzian submanifolds

The class $\mathscr{M}$ is neither contained nor contains the class of Lipschitzian submanifolds of $\mathbb{R}^n$. Example 8, paragraph 13 of [1] gives a Lipschitizian submanifold of $\mathbb{R}^n$ with a boundary, of full dimension, which does not belong to $\mathscr{M}$. Conversely, the subset $M = [-1, 1] \times \{0\} \cup \{0\} \times [-1, 1]$ of $\mathbb{R}^2$ belongs to $\mathscr{M}$ but is not a submanifold of $\mathbb{R}^n$.

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