Approximation methods in equilibrium and fixed point theory

Proximally nondegenerate sets

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Abstract. The aim of this paper is to show how approximation methods allow to extend the Equilibrium and Fixed Point Theory to the nonsmooth and nonconvex setting. Our approach relies heavily on the use of the distance function d_M to a set M. We show that, for our purpose, Clarke's subdifferential ∂d_M is too big, and that the appropriate notion is the upper limit $\partial_+ d_M(\overline{x}) \overset{def}{=} \limsup_{x\to\overline{x}, x\notin M} \partial d_M(x)$.

Keywords: Approximation, Generalized Equilibria, proximally nondegenerate sets, epi-Lipschitzian sets, Clarke's normal cone

1 Introduction[3]

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[3] We let $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$ and sgn $x = x/|x|$ if $x \in \mathbb{R} \setminus \{0\}$. If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ belong to $\mathbb{R}^n$, we denote $(x|y) = \sum_{i=1}^n x_i y_i$, the scalar product of $\mathbb{R}^n$, $\|x\| = \sqrt{(x|x)}$, the Euclidean norm; we denote $B(x,r) = \{y \in \mathbb{R}^n | \ \|x-y\| < r\}$, $\overline{B}(x,r) = \{y \in \mathbb{R}^n | \ \|x-y\| \leq r\}$, $S(x,r) = \{y \in \mathbb{R}^n | \ \|x-y\| = r\}$, and $S = S(0,1)$. If $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^n$, and $x \in \mathbb{R}^n$, we let $d_X(x) = \inf_{y\in X} \|x-y\|$, $X \setminus Y = \{x \in X | x \notin Y\}$ the set-difference of the sets X and Y, $X + Y = \{x+y | x \in X, y \in Y\}$, the sum of the sets X and Y, $B(X,r) = X + B(0,r)$, $\overline{B}(X,r) = X + \overline{B}(0,r)$, cl$X$, the closure of X, int$X$, the interior of X, bd$X$ =cl$X \setminus$int$X$, the boundary of X, $X^\circ = \{y \in \mathbb{R}^n | \forall x \in X, (y|x) \leq 0\}$, the negative polar cone of X, co$X$, the convex hull of X. A map $f : X \to \mathbb{R}$ is locally Lipschitzian if, for every $x \in X$, there is $\varepsilon > 0$ and $L > 0$ such that $\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\|$ for every x_1 and x_2 in $B(x,\varepsilon)$. A correspondence F from X to $\mathbb{R}^n$ is a map from X to the set of all the subsets of $\mathbb{R}^n$; the correspondence F is said to be upper semicontinuous (u.s.c.), resp. lower semicontinuous (l.s.c.), if the set $\{x \in X | F(x) \subset V\}$, resp. the set $\{x \in X | F(x) \cap V \neq \emptyset\}$, is open in X for every open set $V \subset \mathbb{R}^n$. If F is a correspondence from X to $\mathbb{R}^n$, its graph, denoted G(F), is defined by G$(F) = \{(x,y) \in X \times \mathbb{R}^n | y \in F(x)\}$.

We first recall some definitions. A correspondence F from M to $\mathbb{R}^n$ is a map from M to the set of all the subsets of $\mathbb{R}^n$. If $X \subset \mathbb{R}^n$ and $\overline{x} \in \mathbb{R}^n$ we define:

$$\limsup_{x \to \overline{x}, x \in X} F(x) = \{v \in \mathbb{R}^n | \exists (x_k) \subset X, \exists (v_k) \subset \mathbb{R}^n, x_k \to \overline{x}, v_k \in F(x_k), v_k \to v\}.$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}^n$, we denote by $\nabla f(x)$ the gradient of f at x. If $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitzian and $\overline{x} \in \mathbb{R}^n$, its Clarke's subdifferential at $\overline{x}$, denoted $\partial f(\overline{x})$, is defined by:

$$\partial f(\overline{x}) = \mathrm{co} \limsup_{x \to \overline{x}, x \in \mathrm{Dom}\,(\nabla f)} \{\nabla f(x)\},$$

where $\mathrm{Dom}\,(\nabla f)$ is the set on which f is differentiable.[4]

Let $M \subset \mathbb{R}^n$ be closed. Noticing that d_M, the distance function to M, is Lipschitzian, from above, one can define, at $x \in M$, Clarke's normal cone $N_M(x)$ by:

$$N_M(x) = \mathrm{cl}\,(\cup_{\lambda \geq 0} \lambda \partial d_M(x)).$$

The aim of this paper is to show how approximation methods allow to extend the Equilibrium and Fixed Point Theory to the nonsmooth and nonconvex setting. Precisely, we look for conditions on a nonempty, compact set $M \subset \mathbb{R}^n$ so that the following assertions hold for $N(x) = N_M(x)$ (Clarke's normal cone to M at x), in which case $N(x)^\circ = T_M(x)$ (Clarke's tangent cone to M at x). In the definitions below of Assertions $(E; N)$ and $(GE; N)$, we only assume that N is a correspondence defined on M, with values in $\mathbb{R}^n$, such that $N(x)$ is a closed cone (of vertex 0), for every $x \in M$. This will allow us later to consider other normal cones smaller than Clarke's one.

Assertion $(E; N)$ **[Equilibria]** *For every u.s.c. correspondence F from M to $\mathbb{R}^n$, with nonempty, convex, compact values, such that $F(x) \cap N(x)^\circ \neq \emptyset$ for every $x \in M$, there exists a N-equilibrium $x^* \in M$ of F in the sense that:*

$$x^* \in M \text{ and } 0 \in F(x^*).$$

Assertion $(GE; N)$ **[Generalized Equilibria]** *For every u.s.c. correspondence F from M to $\mathbb{R}^n$, with nonempty, convex, compact values, there exists a generalized N-equilibrium $x^* \in M$ of F in the sense that:*

$$x^* \in M \text{ and } 0 \in F(x^*) - N(x^*).$$

The relations between the assertions $(E; N)$ and $(GE; N)$ are extensively studied in Section 5.

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[4] We recall that $\mathbb{R}^n \setminus \mathrm{Dom}\,(\nabla f)$ is of Lebesgue measure zero, from Rademacher's theorem.

The convex and smooth case

The assertions $(E; N)$ and $(GE; N)$ are respectively deduced from Brouwer-Kakutani and Poincaré-Hopf theorems if M is convex or smooth, as recalled below.

Let $M \subset \mathbb{R}^n$ be nonempty, compact, and convex, let F be a u.s.c. correspondence from M to $\mathbb{R}^n$, with nonempty, convex, compact values. Then:

$$\begin{array}{ll}
[F(M) \subset M] & \Rightarrow [\exists x^* \in M,\ x^* \in F(x^*)]; \\
[\forall x \in M, F(x) \cap T_M(x) \neq \emptyset] & \Rightarrow [\exists x^* \in M,\ 0 \in F(x^*)]; \\
[\text{no condition on } F] & [\exists x^* \in M,\ 0 \in F(x^*) - N_M(x^*)].
\end{array}$$

Let $M \subset \mathbb{R}^n$ be nonempty, compact, and smooth[5]. Following Milnor [**?**], the Euler characteristic $\chi(M)$ of M is an integer equal to:

$$\chi(M) = \deg(g_M, \operatorname{int} M, 0),$$

where $g_M = N_M \cap S$ is the Gauss map of M. See the examples in $\mathbb{R}^2$ illustrated in Fig. 1.

Fig. 1. $\chi(M_1) = 1$, $\chi(M_2) = 0$, $\chi(M_3) = -1$

Then $\chi(M) = \deg(F, \operatorname{int} M, 0)$, if F points outward, i.e., $F(x) \cap -T_M(x) \neq \emptyset$ and $0 \notin F(x)$, for every $x \in \operatorname{bd} M$. Hence one deduces that:

(GE) $[\chi(M) \neq 0$ and no condition on $F] \Rightarrow [\exists x^* \in M, 0 \in F(x^*) - N_M(x^*)]$.

2 Existence of (generalized) equilibria

A natural idea to obtain further results in the nonsmooth and nonconvex setting is to approximate a set M by a sequence of smooth sets or by a sequence of convex sets. Convex sets are not good candidates for approximating a set M, since the set $M = \liminf M_k$ is convex if the sets M_k are convex. So we will use hereafter the smooth approximation.

The approximation approach of this paper can be summarized by the following result. We recall that, if (M_k) is a sequence of subsets of $\mathbb{R}^n$, then:

$$\limsup M_k = \{x \in \mathbb{R}^n | \exists (x_k) \subset \mathbb{R}^n, \exists \varphi \in \mathcal{I}, x_k \to x, x_k \in M_{\varphi(k)} \text{ for all } k\},$$

where $\mathcal{I}$ is the set of all increasing maps $\varphi : \mathbb{N} \to \mathbb{N}$.

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[5] i.e., a C^2 submanifold of $\mathbb{R}^n$, with a boundary and of full dimension.

Theorem 1 ([?]). *Let* $M \subset \mathbb{R}^n$ *be compact, admitting a smooth normal approximation in the sense that there is a sequence* (M_k) *of closed subsets of* $\mathbb{R}^n$ *such that:*

(sc) $M = \limsup_{k\to\infty} M_k$*;*

(s) M_k *is smooth and* $M_k \subset B(M,1)$*, for every* $k \in \mathbb{N}$*;*

(nc) $\limsup_{k\to\infty} \mathrm{G}(N_{M_k}) \subset \mathrm{G}(N_M)$*.*

Assume in addition that:

(χ_k) $\chi(M_k) \neq 0$ *for every* $k \in \mathbb{N}$*;*

Then the assertions $(E; N_M)$ *and* $(GE; N_M)$ *hold.*

Proof. **Assertion** $(GE; N_M)$**.** Let F be an u.s.c. correspondence from M to $\mathbb{R}^n$, with nonempty, convex, compact values. For every $k \in \mathbb{N}$, since M_k is smooth and $\chi(M_k) \neq 0$, there is a generalized equilibrium x_k of F on M_k:

$$x_k \in M_k \text{ and } 0 \in F(x_k) - N_{M_k}(x_k),$$

i.e., there is $y_k \in F(x_k)$ such that $(x_k, y_k) \in \mathrm{G}(N_{M_k})$. Without any loss of generality, by a compactness argument, there is (x^*, y^*) such that:

$$\begin{aligned}(x^*, y^*) &= \lim(x_k, y_k) \in \limsup \mathrm{G}(N_{M_k}) \subset \mathrm{G}(N_M),\\ (x^*, y^*) &= \lim(x_k, y_k) \in \mathrm{cl}\,[\mathrm{G}(F)] = \mathrm{G}(F),\end{aligned}$$

Hence $x^* \in M$ and $0 \in F(x^*) - N_M(x^*)$.

Assertion $(E; N_M)$**.** First define the correspondence $\limsup N_{M_k}$ on the set M by:

$$(\limsup N_{M_k})(x) = \limsup_{k\to\infty, x'\to x} N_{M_k}(x').$$

Note that we have shown above that the assertion $(GE; \limsup_{k\to\infty} N_{M_k})$ holds.The implication:

$$(GE; \limsup_{k\to\infty} N_{M_k}) \Rightarrow (E; \limsup_{k\to\infty} N_{M_k})$$

is a consequence of Lemma 1 proved in Section 5, noticing that the correspondence $x \mapsto \limsup_{k\to\infty} N_{M_k}(x)$ has a closed graph. From the inclusion $\limsup_{k\to\infty} \mathrm{G}(N_{M_k}) \subset \mathrm{G}(N_M)$, we deduce that:

$$\forall x \in M, (\limsup N_{M_k})(x) \subset N_M(x),$$

hence the implication $[(E; \limsup_{k\to\infty} N_{M_k}) \Rightarrow (E; N_M)]$ holds. $\square$

Remark 1. Theorem 1 may not hold without the assumption (nc). Indeed, consider, in $\mathbb{R}^2$:

$$\begin{aligned}M &= \overline{B}(0,1) \setminus B(0,1/2);\\ M_k &= M \cup \overline{B}((0, 1 + \tfrac{1}{2^k}), \tfrac{1}{2^{k+1}}).\end{aligned}$$

The sequence (M_k) satisfies the assumptions (s) (for $k \geq 1$), (χ_k) and (sc) of Theorem 1, but the assumption (nc) is not satisfied. In this case both assertions $(E; N_M)$ and $(GE; N_M)$ do not hold.

Theorem 1 is very general since many classes of sets do satisfy its assumptions, but the main drawback of these assumptions is that they are not expressed in terms of the set M and do depend upon the chosen approximation (M_k). Thus the following questions arise:

Question 1. Can we replace the previous topological assumption:

$$(\chi_k) \quad \chi(M_k) \neq 0 \text{ for every } k \in \mathbb{N},$$

by an intrinsic topological assumption on M? For example, can we replace it by the assumption $\chi(M) \neq 0$?

Recall that, in general, the Euler characteristic of M is defined by (see [**?**]):

$$\chi(M) = \sum_{k \in \mathbb{N}} (-1)^k \mathrm{rg} H_k M,$$

where $(H_k M)_{k \in \mathbb{N}}$ are the (singular) homology groups of M, whenever the sum has a meaning.

A first answer is given by the two following examples in which $\chi(M)$, the Euler characteristic of M, is not defined:

Example 1. In $\mathbb{R}^2$, take $M = \{0\} \cup \{(\frac{1}{2^n}, 0) | n \in \mathbb{N}\}$ (Fig. 2) . Then M clearly satisfies the assumptions of Theorem 1, with an approximation (M_k) such that $\chi(M_k) = k$, but the Euler characteristic of M is not defined.

Fig. 2. $M = \{0\} \cup \{(\frac{1}{2^n}, 0) | n \in \mathbb{N}\}$

Example 2. In $\mathbb{R}^2$, take $M = \cup_{n \in \mathbb{N}} S\Big((\frac{1}{2^n}, 0), \frac{1}{2^n}\Big)$, which is arc-wise connected (Fig. 3). Then M clearly satisfies the assumptions of Theorem 1, with an approximation (M_k) such that $\chi(M_k) = -k$, but the Euler characteristic of M is not defined.

Fig. 3. $M = \cup_{n \in \mathbb{N}} S\Big((\frac{1}{2^n}, 0), \frac{1}{2^n}\Big)$

Question 2. Which classes of sets satisfy the assumptions of Theorem 1, with possibly only intrinsic assumptions on the set M?

Since we want that the same topological properties hold for M and M_k (at least to get the equality $\chi(M) = \chi(M_k)$, which is the case in particular if M and M_k are homeomorphic), a natural idea is to write:

$$M = \{x \in \mathbb{R}^n | f_M(x) \leq 0\} \text{ and } M_k = \{x \in \mathbb{R}^n | f_M(x) \leq 1/k\},$$

with a "good" function f_M (say locally Lipschitzian) which also represents the normal cone N_M in the sense that $N_M(x) = \mathrm{cl}\,(\cup_{\lambda \geq 0} \lambda \partial f_M(x))$ (as for $f_M = d_M$).

First try: the distance function d_M. But the following non-degeneracy condition, which seems essential to relate the topological properties of the sets M and M_k:

$$(nd) \quad 0 \notin \partial f(x) \text{ if } f(x) = 0,$$

is never satisfied with $f = d_M$.

Second try: the function:

$$\Delta_M = d_M - d_{\mathbb{R}^n \setminus M}.$$

(introduced and studied in optimization by Hiriart-Urruty [**?**] and [**?**])
If the corresponding nondegeneracy condition:

$$0 \notin \partial \Delta_M(x) \text{ if } \Delta_M(x) = 0$$

is satisfied, then the set M is epi-Lipschitzian, in the sense that its Clarke's normal cone $N_M(x)$ is pointed (i.e., if $N_M(x) \cap -N_M(x) = \{0\}$) at every $x \in M$.[6]

In fact, a closed subset M of $\mathbb{R}^n$ is epi-Lipschitzian if and only if there is a Lipschitzian function $f : \mathbb{R}^n \to \mathbb{R}$ such that $M = \{x \in \mathbb{R}^n | f(x) \leq 0\}$ and such that (nd) $0 \notin \partial f(x)$ if $f(x) = 0$. The sufficient part is classical and the necessary part follows from Proposition 1 below. This justifies that we devote the next section to the epi-Lipschitzian case.

3 The epi-Lipschitzian case

The class of epi-Lipschitzian sets, introduced in optimization by Rockafellar, is of particular importance since it includes both (i) closed convex sets

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[6] The above definition is not the original one introduced by Rockafellar [**?**], but is equivalent to it. Originally, Rockafellar defined epi-Lipschitzian sets as sets that can be locally written as the epigraph of a Lipschitzian function.

with a nonempty interior, and (*ii*) sets defined by finite smooth inequality constraints satisfying a nondegeneracy assumption (independence of the gradients of the binding constraints).

3.1 Representation by $\varDelta_M$

The function $\varDelta_M$ provides a first way to "represent" the set M, and N_M, by a Lipschitzian function.

Proposition 1 (Lipschitzian representation, [?]). *Let $M \subset \mathbb{R}^n$ be closed such that $M \neq \emptyset$ and $M \neq \mathbb{R}^n$, then the three following conditions are equivalent:*
(i) M is epi-Lipschitzian;
(ii) for every $x \in \operatorname{bd} M$, $0 \notin \partial \varDelta_M(x)$;
(iii) $\varDelta_M$ is a Lipschitzian representation of M, in the following sense:
  - (l) *$\varDelta_M$ is Lipschitzian on $\mathbb{R}^n$;*
  - (r) *$M = \{x \in \mathbb{R}^n | \varDelta_M(x) \leq 0\}$;*
  - (nr) *$N_M(x) = \cup_{\lambda \geq 0} \lambda \partial \varDelta_M(x)$ for every $x \in \operatorname{bd} M$;*
  - (nd) *$0 \notin \partial \varDelta_M(x)$ if $\varDelta_M(x) = 0$.*

Moreover, if (i), (ii), (iii) holds, then: $\forall x \in \operatorname{bd} M, \partial \varDelta_M(x) \subset \operatorname{co}(N_M(x) \cap S)$.

For the proof of Proposition 1, we refer to [?]. In general, the function $\varDelta_M$ is **not differentiable**. Thus the sets:

$$M_k = \{x \in \mathbb{R}^n | \varDelta_M(x) \leq 1/k\}$$

may not be smooth, even though they are epi-Lipschitzian if M is epi-Lipschitzian. To recover smoothness, we shall need to "smooth" $\varDelta_M$ with a convolution type argument.

3.2 Quasi-smooth representation

Theorem 2 (quasi-smooth representation, [?]). *Let $M \subset \mathbb{R}^n$ be closed, then the two following conditions are equivalent:*
(a) M is epi-Lipschitzian;
(b) there is a Lipschitzian map $f_M : \mathbb{R}^n \to \mathbb{R}$ such that:
  - (s) *f_M is C^∞ on $\mathbb{R}^n \setminus \operatorname{bd} M$;*
  - (r) *$M = \{x \in \mathbb{R}^n | f_M(x) \leq 0\}$ and $f_M^{-1}((-\infty, 1/2]) \subset B(M, 1)$;*
  - (nd) *$0 \notin \partial f_M(x)$ if $f_M(x) = 0$;*
  - $(\partial - nr)$ *$\partial f_M(x) = \partial \varDelta_M(x)$ if $f_M(x) = 0$;*
  - (nr) *$N_M(x) = \cup_{\lambda \geq 0} \lambda \partial f_M(x)$ for every $x \in \operatorname{bd} M$.*

The implication $[(b) \Rightarrow (a)]$ is classical (see Clarke [?]). We now sketch the proof of the implication $[(a) \Rightarrow (b)]$. For the details, we refer to [?].
Idea of the proof of the implication $[(a) \Rightarrow (b)]$: "smoothing" $\varDelta_M$.

From Proposition 1, the function Δ_M satisfies all the assertions of (b) but the smoothness one. We now "smooth" Δ_M as follows. Let $\rho : \mathbb{R}^n \to \mathbb{R}_+$ be a given C^1 function. We define the function $f_\rho : \mathbb{R}^n \to \mathbb{R}$ by:

$$f_\rho(x) = \int_{\overline{B}(0,1)} \theta(t) \Delta_M(x - \rho(x)t)dt, \text{ for every } x \in \mathbb{R}^n,$$

where θ is a mollifier. A first idea is to take $\rho = |\Delta_M|$, but the corresponding function $f_{|\Delta_M|}$ may not be smooth in general. Then the proof of $[(a) \Rightarrow (b)]$ goes in two steps.

Step 1. We smooth $|\Delta_M|$. Precisely, we show that there is a C^1 function $\rho : \mathbb{R}^n \to \mathbb{R}$ such that $0 \leq \rho \leq |\Delta_M|$, $\|\nabla \rho\| \leq |\Delta_M|$, and $\rho(x) = 0$ if and only if $\Delta_M(x) = 0$.

Step 2. We show that $f_M = f_\rho$ satisfies the assertion (b) if in the above step 1, we consider (instead of $|\Delta_M|$) the map $\Delta = \min\{1, \Delta_M^2, \frac{1}{2}|\Delta_M|\}$.

3.3 Smooth normal approximation

Let $M \subset \mathbb{R}^n$ be compact and epi-Lipschitzian, and let $f_M : \mathbb{R}^n \to \mathbb{R}$ be given by Theorem 2. Then the sets:

$$M_k = \{x \in \mathbb{R}^n | f_M(x) \leq 1/k\}$$

define a smooth normal external approximation[7] of M in the following sense (stronger that the one in Theorem 1):
(sc) $M = \cap_{k \geq 1} M_k$ and $M_{k+1} \subset \operatorname{int} M_k$ for every $k \geq 1$;
(s) M_k is smooth and $M_k \subset B(M, 1)$ for every $k \geq 1$;
(nc) $\limsup_{k \to \infty} \mathrm{G}(N_{M_k}) \subset \mathrm{G}(N_M)$;
and furthermore they satisfy the following topological property:
(*homeo*) M and M_k are homeomorphic for every $k \geq 1$.

The function f_M allows us to construct a retraction of M_k on M, by following the vector field $dx/dt = -\nabla f_M(x(t))$, as illustrated in Fig. 4.

Fig. 4.

By slightly perturbing the vector field ∇f_M, one moreover shows (Bonnisseau and Cornet [**?**]) that the sets M and M_k are homeomorphic. This result precises previous results by relating the topological properties of M and its approximation.

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[7] In the same way, the sets $M^k = \{x \in \mathbb{R}^n | f_M(x) \leq -1/k\}$ define a smooth normal internal approximation of M. We refer to [**?**] for a general study of normal approximations of epi-Lipschitzian sets

Other results in the literature

There are stated without the condition (nc), and without topological properties of the approximation (Nečas [**?**], Massari and Pepe [**?**], Doktor [**?**], Benoist [**?**], Clarke, Ledyaev & Stern [**?**]).

3.4 A necessary and sufficient condition for the existence of equilibria

In the epi-Lipschitzian case, Theorem 1 can be precised, and we provide a necessary and sufficient condition.

Theorem 3. *Let $M \subset \mathbb{R}^n$ be nonempty, compact and epi-Lipschitzian. Then the assertions $(E; N_M)$ and $(GE; N_M)$ (and to their single-valued versions) are equivalent to the following condition:*

Assertion (χ)*: There is a connected component M_i of M such that $\chi(M_i) \neq 0$.*

Proof. **Sufficient part.** Assume first that M is connected; then $\chi(M) \neq 0$. Let (M_k) be a smooth normal approximation of M. Since M_k is homeomorphic to M, $\chi(M_k) \neq 0$ and the assumptions of Theorem 1 are satisfied, hence the assertions $(E; N_M)$ and $(GE; N_M)$ are satisfied. To get the result in the case where M is not connected, one only needs to check that M has a finite number of epi-Lipschitzian connected components and that $\chi(M) = \sum \chi(M_i)$ (see [**?**]).

Necessary part. It relies on the existence of a smooth internal approximation of M and on the corresponding classical result for smooth sets (see, for example, Milnor [**?**]). We refer to [**?**]. □

Remark 2, on Assertion (χ) : Assertion (χ) is satisfied if $\chi(M) \neq 0$ (since $\chi(M) = \sum_{i \in I} \chi(M_i)$), but it cannot be replaced by the assertion $\chi(M) \neq 0$. Take $M = M_1 \cup M_2$, with M_1 and M_2 smooth and connected, $M_1 \cap M_2 = \emptyset$, $\chi(M_1) = 1$, and $\chi(M_2) = -1$. Then $\chi(M) = 0$, and the assertions $(E; N_M)$ and $(GE; N_M)$ hold true.

Other results in the literature

In these results, we assume that $M \subset \mathbb{R}^n$ is nonempty, compact and epi-Lipschitzian.

Theorem [Cornet [**?**]] $(\chi(M) \neq 0) \Rightarrow (E; N_M) \Rightarrow (GE; N_M)$.

Since $(\chi(M) \neq 0) \Rightarrow (\chi)$, it is a consequence of Theorem 3.

Theorem [Clarke, Ledyaev, Stern [?]] *Assume that M is homeomorphic to a compact convex subset of $\mathbb{R}^n$. Then Assertion $(E; N_M)$ holds true.*

This result is a consequence of Theorem 3 and of the fact that $\chi(M) = 1$ if M is homeomorphic to a compact convex subset of $\mathbb{R}^n$. [?] shows another existence result in the more general setting of infinite dimension.

4 The proximally nondegenerate case

In this section, we introduce a class of sets, wider than the class of epi-Lipschitzian sets, which also satisfies the assumptions of Theorem 1, with intrinsic conditions on the set M. For the proofs of the results contained in this section, we refer to [?] and [?].

4.1 Definition and examples

Noticing that, at points $x \in \operatorname{bd} M$, Clarke's subdifferential of the distance function d_M is always "too big", since:

$$0 \in \partial d_M(x),$$

we define for $x \in \operatorname{bd} M$ (this notion can be defined on $\mathbb{R}^n$, see [?]) the set:

$$\partial_+ d_M(x) = \limsup_{x' \to x,\ x' \notin M} \partial d_M(x'),$$

which is clearly contained in $\partial d_M(x)$ (from the u.s.c. of the correspondence ∂d_M). The following proposition gives a characterization of epi-Lipschitzian sets in terms of this new notion.

Proposition 2 ([?]). *A closed subset M of $\mathbb{R}^n$ is epi-Lipschitzian if and only if:*

$$0 \notin \operatorname{co} \partial_+ d_M(x), \text{ for every } x \in \operatorname{bd} M.$$

It is then natural to extend the class of epi-Lipschitzian sets by replacing the set $\operatorname{co} \partial_+ d_M(x)$ with the set $\partial_+ d_M(x)$ as it is done in the following definition.

Definition 1 ([?]). A closed subset M of $\mathbb{R}^n$ is proximally nondegenerate if, at every $x \in M$, one of the following equivalent assertions is satisfied:

$$0 \notin \partial_+ d_M(x); \tag{1a}$$

$$\exists \alpha > 0, \forall x' \in B(x, \alpha) \setminus M, \forall p \in \partial d_M(x'), \|p\| \geq \alpha. \tag{1b}$$

The class of proximally nondegenerate sets is very broad, as shown by the following proposition.

Proposition 3 ([?]). *A closed set $M \subset \mathbb{R}^n$ is proximally nondegenerate if it satisfies one of the following conditions:*

(i) *M is convex;*
(ii) *$\mathbb{R}^n \setminus M$ is convex;*
(iii) *M is a C^1 submanifold of $\mathbb{R}^n$, with or without a boundary, with or without corners;*
(iv) *M is epi-Lipschitzian;*
(v) *M is proximally smooth (Clarke, Stern, and Wolenski [?]);*
(vi) *M is proximally regular (Poliquin, Rockafellar, Thibault, [?] and [?])*

4.2 Quasi-smooth representation and smooth approximation

We now smooth the distance function to a set M (in the same way as we did for Δ_M, in Theorem 2), to get a quasi-smooth representation of the set M, as precisely stated in the following result.

Theorem 4 ([?]). *Let M be a closed subset of $\mathbb{R}^n$, then there is a Lipschitzian function $f_M : \mathbb{R}^n \to \mathbb{R}_+$ such that:*

(a) *f_M is C^∞ on $\mathbb{R}^n \setminus M$;*
(b) *$M = \{x \in \mathbb{R}^n | f_M(x) = 0\}$, and $f_M^{-1}((0, 1/2]) \subset B(\mathrm{bd}\, M, 1)$;*
(c) *$\forall x \notin M$, $\|\nabla f_M(x)\| \geq (1/2) \min\{\|v\| \, | v \in \partial d_M(\overline{B}(x, d_M(x)/2))\}$;*
(d) *$\forall x \in M$, $\partial_+ f_M(x) \subset \partial_+ d_M(x)$.*

Let $M \subset \mathbb{R}^n$ be compact and proximally nondegenerate, and let $f_M : \mathbb{R}^n \to \mathbb{R}$ be given by Theorem 4. Then the sets:

$$M_k = \{x \in \mathbb{R}^n | f_M(x) \leq 1/k\}$$

define a smooth normal external approximation of M in the same sense as before, and they satisfy the weaker topological property:

(ret) for every $k \geq 1$, M is a deformation retract of M_k,

i.e., there is a continuous map $H : [0, 1] \times M_k \to M_k$ such that, for every $x \in M_k$, $H(0, x) = x$, and $H(1, x) \in M$, and, for every $x \in M$, $H(1, x) = x$. We note that in general, the sets M and M_k are not homeomorphic.

4.3 Existence of equilibria

In the proximally nondegenerate case, Theorem 1 can be precised and we get the following result, stated intrinsically in terms of the set M.

Theorem 5 ([?]). *Let $M \subset \mathbb{R}^n$ be nonempty, compact, proximally nondegenerate, and assume* **Assertion** *(χ). Then the assertions $(E; N_M)$, $(GE; N_M)$, and $(GE; \widetilde{N}_M)$ hold, where $\widetilde{N}_M(x) = \cup_{\lambda \geq 0} \lambda \partial_+ d_M(x)$ for every $x \in M$.*

We point out that the cone $\widetilde{N}_M(x)$ is smaller than Clarke's normal cone $N_M(x)$, and may not be convex (Fig.5).

Fig. 5. $M = [-1,0] \times \{0\} \cup \{0\} \times [-1,0]$, $\widetilde{N}_M(0) = \mathbb{R}^2_+ \cup \mathbb{R}^2_-$, $N_M(0) = \mathbb{R}^2$.

Proof. Assume first that M is connected; then $\chi(M) \neq 0$. Let M_k be a smooth normal approximation of M. Since M is a deformation retract of M_k, $\chi(M_k) = \chi(M) \neq 0$ and the assumptions of Theorem 1 are satisfied, hence the assertions $(E; N_M)$ and $(GE; N_M)$ are satisfied. Assertion $(GE; \widetilde{N}_M)$ is a consequence of the inclusion:

$$\limsup_{k \to \infty} \mathrm{G}(N_{M_k}) \subset \mathrm{G}(\widetilde{N}_M).$$

To get the result in the case where M is not connected, one only need to check that M has a finite number of connected component and that $\chi(M) = \sum \chi(M_i)$ (see [**?**]). $\square$

Remark 3. Theorem 5 may no longer be true if the set M is not proximally nondegenerate. Consider the following connected continuous submanifold with a boundary, illustrated in Figure **??**:

$$M = \left\{ (x,y) \in \mathbb{R}^2 \middle| \begin{array}{l} |x| \geq 1/4,\ (x - (1/4)\mathrm{sgn}\, x)^2 + (y - \mathrm{sgn}\, y)^2 \in [1/4, 1], \\ \text{or } |x| < 1/4,\ y \in [-2, -3/2] \cup [-1/2, 1/2] \cup [3/2, 2] \end{array} \right\}.$$

Assertion (χ) holds true, since $\chi(M) = -1$. But, since there is a continuous map $f : M \to \mathbb{R}^2$, such that $f(x) \in T_M(x) \setminus \{0\}$ for every $x \in M$, the assertions $(GE; N_M)$ and $(E; N_M)$ do not hold.

Fig. 6.

Remark 4. In Theorem 5, Assertion (χ) is sufficient, but not necessary, for the assertions $(GE; N_M)$ and $(E; N_M)$ to hold. In $\mathbb{R}^2$, consider the square:

$$M = \{(x,y) \in \mathbb{R}^2 |\, \sup\{|x|, |y|\} = 1\}.$$

Then M is proximally nondegenerate. Assertion (χ) does not hold, since M is connected and $\chi(M) = 0$. the assertions $(GE; N_M)$ and $(E; N_M)$ hold, since $N_M(1,1) = \mathbb{R}^2$ and $T_M(1,1) = \{0\}$.

Other results in the literature

Ben-El-Mechaiekh and Kryszewski [**?**] show the implication $[(\chi(M) \neq 0) \Rightarrow (E; N_M)]$ in infinite dimension, for the wider class of $\mathcal{L}$-retracts. Ćwiszewski and Kryszewski [**?**] show the implication $[(\chi(M) \neq 0) \Rightarrow (GE; N_M)]$ for the class of $\mathcal{L}$-retracts. Both papers use the Čech cohomology and their key-tool is Lefschetz fixed-point theorem.

Remark 5. One could think of enlarging the class of proximally nondegenerate sets by considering the class $\mathcal{M}$ of closed subsets M of $\mathbb{R}^n$ such that:

$$\forall x \in M,\ \exists \alpha > 0,\ \forall x' \in B(x,\alpha) \setminus M, 0 \notin \partial d_M(x')\ (1a').$$

But Theorem 5 may not be true for sets in $\mathcal{M}$. In $\mathbb{R}^2$, consider the spiral defined by:

$$M = S(0,1) \cup \{((1+1/\theta)\cos\theta, (1+1/\theta)\sin\theta) | \theta \in [3,\infty)\},$$

and the tangent field $f : M \to \mathbb{R}^2$ defined by $f((1+1/\theta)\cos\theta, (1+1/\theta)\sin\theta) = (-\sin\theta - \cos\theta/\theta^2, \cos\theta - \sin\theta/\theta^2)$ and $f(\cos\theta, \sin\theta) = (-\sin\theta, \cos\theta)$. Then $M \in \mathcal{M}$, $\chi(M) = 1$, and $f(x) \in T_M(x) \setminus \{0\}$ for every $x \in M$, hence none of the assertions $(GE; \widetilde{N}_M)$, $(GE; N_M)$, $(E; N_M)$ is satisfied. Notice also that M is not a neighborhood retract.

5 Relations between $(GE; N)$ and $(E; N)$

In this section, we state and prove a lemma which is a keytool in the proof of Theorem 1. This lemma precises the relations between the assertions $(GE; N)$ and $(E; N)$, and their single-valued versions, denoted respectively $(GE_{sv}; N)$, and $(E_{sv}; N)$.

Lemma 1. *Let $M \subset \mathbb{R}^n$ be a compact set, let N be a correspondence from M to $\mathbb{R}^n$, such that $N(x)$ is a cone for every $x \in M$:*
(a) assume that N has a closed graph, then:

$$(GE_{sv}; N) \Leftrightarrow (GE; N) \Rightarrow (E; N) \Rightarrow (E_{sv}; N);$$

(b) additionally assume that N has convex values, then:

$$(GE; N) \Leftrightarrow (GE_{sv}; N) \Leftrightarrow (E; N) \Leftrightarrow (E_{sv}; N).$$

Remark 6. If the correspondence N does not have convex values, then the implication $[(E; N) \Rightarrow (GE; N)]$. See Remarks 10 and 11 below, with $N = \widehat{N}_M$, the limiting normal cone.

Remark 7. If $\operatorname{co} N$ does not have a closed graph (even if N has a closed graph), the implication $[(E_{sv}; N) \Rightarrow (E; N)](iv)$ may not be true. In $\mathbb{R}^3$, consider $M = M_1 \cup M_2 \cup \{0\}$, with $M_1 = (0,1] \times \{0\} \times \{0\}$, $M_2 = \{0\} \times (0,1] \times \{0\}$, define the correspondence N by $N(0) = \mathbb{R}(0,0,1)$, $N((x,0,0)) = \mathbb{R}(x,0,1) \cup \mathbb{R}(-x,0,1)$ for $(x,0,0) \in M_1$ and $N((0,y,0)) = \mathbb{R}(0,y,1) \cup \mathbb{R}(0,-y,1)$ for $(0,y,0) \in M_2$. Then $(E_{sv}; N)$ holds, but $(E; N)$ does not hold (consider the correspondence F defined by $F(0) = \operatorname{co}\{(0,1,0),(1,0,0)\}$, $F(x) = (0,1,0)$ for $x \in M_1$ and $F(x) = (1,0,0)$ for $x \in M_2$.

Before proving Lemma **??**, we shall need a claim, the proof of which is (more or less) classical , and is left to the reader.

Claim (1). Let F be a correspondence from M to $\mathbb{R}^n$, and let T be a l.s.c. correspondence from M to $\mathbb{R}^n$, with convex values, such that $F(x) \cap T(x) \neq \emptyset$ for every $x \in M$. We let, for $x \in M$ and $k \in \mathbb{N} \setminus \{0\}$:

$$F_k(x) = \operatorname{co} B\Big(F(B(x, 1/k) \cap M), 1/k\Big).$$

(a) The correspondence $F_k \cap T$ is l.s.c., with nonempty convex values, hence admits a continuous selection (from Michael's selection theorem).
(b) Let (x_k, y_k) be a sequence in $M \times \mathbb{R}^n$ converging to $(x,y) \in M \times \mathbb{R}^n$, such that $y_k \in F_k(x_k)$ for every k. Assume that F is bounded on a neighborhood of x, then $(x,y) \in \operatorname{co}(\limsup(F)(x))$

We now come back to the proof of Lemma **??**.

Proof. Assertion (c) is a clear consequence of the assertions (a) and (b).

Part (a). The implications $[(GE; N) \Rightarrow (GE_{sv}; N)]$ and $[(E; N) \Rightarrow (E_{sv}; N)]$ are immediate.

$[(GE_{sv}; N) \Rightarrow (GE; N)]$. Let F be a u.s.c. correspondence from M to $\mathbb{R}^n$, with convex compact values. For $k \in \mathbb{N} \setminus \{0\}$, the correspondence F_k is defined as above. From Claim 1 (taking $T(x) = \mathbb{R}^n$), the correspondence F_k admits a continuous selection $f_k : M \to \mathbb{R}^n$, i.e., such that:

$$\forall x \in M, f_k(x) \in F_k(x).$$

From $(GE_{sv}; N)$ there is $x_k \in M$ such that $f_k(x_k) \in N(x_k)$. Without any loss of generality, we may assume that the sequence $(x_k, f_k(x_k))$ converges to some (x^*, y^*) in the compact set $M \times \operatorname{cl}\operatorname{co} B(F(M), 1)$. Since the correspondence N has a closed graph, we get that $y^* \in N(x^*)$. Since F is u.s.c., with compact convex values, applying Claim 1 to the sequence $(x_k, f_k(x_k))$, we get that $y^* \in \operatorname{co}(\limsup F)(x^*) \subset F(x^*)$, hence that $0 \in F(x^*) - N(x^*)$, i.e., x^* is a generalized equilibrium of F, hence $(GE; N)$ holds..

[$(GE_{sv}; N) \Rightarrow (E; N)$]. Let F be an u.s.c. correspondence from M to $\mathbb{R}^n$, with convex compact values, such that $F(x) \cap N(x)^\circ \neq \emptyset$ for every $x \in M$. For $k \in \mathbb{N} \setminus \{0\}$, the correspondence F_k is defined as above, and we let, for every $x \in M$:

$$T_k(x) = \{y \in \mathbb{R}^n | \forall p \in N(x) \cap S,\ (y|p) < 1/k\}.$$

Noticing that, for all x, $\emptyset \neq F(x) \cap N(x)^\circ \subset F(x) \cap T_k(x)$, and that the correspondence T_k has convex values, and is lower semicontinuous, since the set $\{x \in M | y \in T_k(x)\}$ is open in M (for its relative topology), for every $y \in \mathbb{R}^n$. From Claim 1, the correspondence $F_k \cap T_k$ admits a continuous selection $f_k : M \to \mathbb{R}^n$, i.e., such that:

$$\forall x \in M, f_k(x) \in F_k(x) \cap T_k(x).$$

From $(GE_{sv}; N)$ there is $x_k \in M$ such that $0 \in f_k(x_k) - N(x_k)$. Without any loss of generality, we may assume that the sequence (x_k) converges to some element $x^* \in M$, and we prove that $(f_k(x_k))$ converges to 0. Indeed, $f_k(x_k) \in N(x_k) \cap T_k(x_k)$, hence $\|f_k(x_k)\| < 1/k$. Applying Claim 1 to the sequence $(x_k, f_k(x_k))$, we get that $0 \in \mathrm{co}\,(\limsup F)(x^*)) \subset F(x^*)$, i.e., x^* is an equilibrium of F, hence $(E; N)$ holds.

Part (b). [$(E_{sv}; N) \Rightarrow (E; N)$]. Let F be an u.s.c. correspondence from M to $\mathbb{R}^n$, with convex compact values, such that $F(x) \cap N(x)^\circ \neq \emptyset$ for every $x \in M$. For $k \in \mathbb{N} \setminus \{0\}$, the correspondence F_k is defined as above. Since the correspondence N has a closed graph with convex values, one easily shows that the correspondence $x \mapsto N(x)^\circ$ is l.s.c.. Hence, from Claim 1, the correspondence $F_k \cap N^\circ$ admits a continuous selection $f_k : M \to \mathbb{R}^n$, i.e., such that:

$$\forall x \in M, f_k(x) \in F_k(x) \cap N(x)^\circ.$$

From $(E_{sv}; N)$ there is $x_k \in M$ such that $f_k(x_k) = 0$. Without any loss of generality, we may assume that the sequence (x_k) converges to some element $x^* \in M$. Applying Claim 1 to the sequence $(x_k, f_k(x_k)) = (x_k, 0)$, we get that $0 \in \mathrm{co}\,(\limsup F)(x^*) \subset F(x^*)$, i.e., x^* is an equilibrium of F, hence $(E; N)$ holds.

[$(E; N) \Rightarrow (GE; N)$]. Let F be an u.s.c. correspondence from M to $\mathbb{R}^n$, with nonempty convex compact values, then F is bounded on the compact set M, i.e., there exists $k > 0$ such that $F(x) \subset \overline{B}(0, k)$ for every $x \in M$. We define the correspondence F from M to $\mathbb{R}^n$ by:

$$\Phi(x) = F(x) - N(x) \cap \overline{B}(0, k).$$

The correspondence Φ is clearly u.s.c. with nonempty convex compact values and we now show that:

$$\forall x \in M, \quad \Phi(x) \cap N(x)^\circ \neq \emptyset.$$

Indeed, let $x \in M$ and let $y \in F(x)$. Since $N(x)$ is a closed convex cone, we recall that there exist a unique element $y_N \in N(x)$ and a unique element $y_T \in N(x)^\circ$ such that $y = y_N + y_T$ and $(y_N | y_T) = 0$. Consequently, the element $y_T = y - y_N$ belongs to $\Phi(x)$ since $\|y_N\|^2 \leq \|y\|^2 \leq k$. Consequently, from Assertion $(E; N)$, there exists $x^* \in M$ such that $0 \in \Phi(x^*) \subset (x^*) - N(x^*)$, i.e., x^* is a generalized equilibrium of F, hence $(GE; N)$ holds. □

Remark 8, on Theorem 3. In the epi-Lipschitzian case, since the correspondence N_M has a closed graph and convex values, Lemma **??** directly (without using Theorem 1) gives the equivalence:

$$(E; N_M) \Leftrightarrow (GE; N_M).$$

Remark 9, on Theorem 5. In the proximally nondegenerate case, the assertions $(GE; N_M)$ and $(E; N_M)$ are both direct consequences of the assertion $(GE; \widetilde{N}_M)$, using Lemma **??**. Indeed, the correspondence $\widetilde{N}_M$ has a closed graph (but, in general, not the correspondence N_M!), hence Lemma **??** gives the implication:

$$(GE; \widetilde{N}_M) \Rightarrow (E; \widetilde{N}_M),$$

and we notice the equivalence $[(E; \widetilde{N}_M) \Leftrightarrow (E; N_M)]$, since $\operatorname{cl}\operatorname{co} \widetilde{N}_M(x) = N_M(x)$. The last implication $[(GE; \widetilde{N}_M) \Rightarrow (GE; N_M)]$ is a direct consequence of the inclusion $\widetilde{N}_M(x) \subset N_M(x)$.

6 Other notions of normal cones

Up to now, we have considered the two cases $N = N_M$ (Clarke's normal cone) and $N = \widetilde{N}_M$. The following remarks discuss the existence problem of equilibria and generalized equilibria, when one considers other notions of normal cones such as $N = \widehat{N}_M$, the limiting normal cone [8] and $N = N_M^B$, Bouligand normal cone. [9]

The case $N = \widehat{N}_M$, the limiting normal cone

Remark 10 (generalized equilibria). The implication $[(\chi) \Rightarrow (GE; \widehat{N}_M)]$ may not be true, even if the set M is compact and epi-Lipschitzian. In $\mathbb{R}^3$, consider the counterexample in Clarke, Ledyaev and Stern [**?**], $\chi(M) = 1$ (for example, M is homeomorphic to a convex set).

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[8] We recall that the limiting normal cone to M at x is defined by $\widehat{N}_M(x) = \cup_{\lambda \geq 0} \lambda \nabla_+ d_M(x) \cup \{0\}$.

[9] We recall that Bouligand normal cone to M at x is defined by $N_M^B(x) = T_M^B(x)^\circ$, where $T_M^B(x) = \{v \in \mathbb{R}^n | \exists (\lambda_k)_{k \in N}, \lambda_k > 0, \exists (y_k)_{k \in \mathbb{N}}, y_k \in M, y_k \to x, v = \lim_{k \to \infty} \lambda_k (y_k - x)\}$.

Remark 11 (equilibria). However, the implication $[(\chi) \Rightarrow (E; \widehat{N}_M)]$ holds under the assumption of Theorem 5, since $\mathrm{cl}\,\mathrm{co}\,\widehat{N}_M(x) = N_M(x)$, which implies the equivalence $[(E; \widehat{N}_M) \Leftrightarrow (E; N_M)]$.

The case $N = N_M^B$, Bouligand normal cone

Remark 12 (generalized equilibria). The implication $[(\chi) \Rightarrow (GE; N_M^B)]$ may not be true, even if the set M is compact and epi-Lipschitzian. Consider the counterexample in [**?**] and note that $N_M^B(0) = \{0\}$.

Remark 13 (equilibria in the multi-valued case). The implication $[(\chi) \Rightarrow (E; N_M^B)]$ may not be true, even if the set M is compact and epi-Lipschitzian. Consider the set M in the counterexample in [**?**].

Remark 14 (equilibria in the single-valued case). The implication $[(\chi) \Rightarrow (E_{sv}; N_M^B)]$ holds under the assumptions of Theorem 5. Indeed, let M be a closed subset of $\mathbb{R}^n$ and let $f : M \to \mathbb{R}^n$ be a continuous map; then from [**?**], the two following assertions are equivalent:
(i) $f(x) \in T_M^B(x)$ for every $x \in M$;
(ii) $f(x) \in T_M(x)$ for every $x \in M$.
Hence, the equivalence $[(E_{sv}; N_M) \Leftrightarrow (E_{sv}; N_M^B)]$ holds.

The case of the upper-limit cone $\limsup N_{M_k}$

Let M be a closed subset of $\mathbb{R}^n$ and let (M_k) be a sequence of closed subsets of $\mathbb{R}^n$ such that $M = \limsup_{k\to\infty} M_k$.

Remark 15. Assume that, for every k, the assertion $(GE; N_{M_k})$ holds. Then the assertions $(GE; \limsup N_{M_k})$ and $(E; \limsup N_{M_k})$ hold. If additionally $\mathrm{G}(\limsup N_{M_k}) \subset \mathrm{G}(N_M)$ (resp. $\mathrm{G}(\limsup N_{M_k}) \subset \mathrm{G}(\widehat{N}_M)$), then $(GE; N)$ and $(E; N)$ hold (resp. $(GE; \widehat{N})$).

Remark 16. Conversely, notice that (see [**?**] and [**?**]):

$$\begin{aligned} &\mathrm{G}(\widehat{N}) \subset \mathrm{G}(\limsup N_{M_k}); \\ &\mathrm{G}(N) \subset \mathrm{G}(\mathrm{cl}\,\mathrm{co}\,\limsup N_{M_k}), \end{aligned}$$[10]

Then we get the implications $[(GE; \widehat{N}_M) \Rightarrow (GE; \limsup N_{M_k})]$ and $[(E; N_M) \Rightarrow (E; \limsup N_{M_k})]$. The converse of these implications may not be true. Consider, in $\mathbb{R}^2$, $M = \overline{B}(0,1) \setminus B(0,1/2)$ and $M_k = M \cup \overline{B}((0, 1 + 1 + \frac{1}{2^k}), \frac{1}{2^{k+1}})$.

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[10] The correspondence $\mathrm{cl}\,\mathrm{co}\,\limsup N_{M_k}$ is defined by: $(\mathrm{cl}\,\mathrm{co}\,\limsup N_{M_k})(x) = \mathrm{cl}\,(\mathrm{co}\,(\limsup N_{M_k}(x)))$.

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