EXISTENCE OF (GENERALIZED) EQUILIBRIA : NECESSARY AND SUFFICIENT CONDITIONS

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Abstract. We provide necessary and sufficient conditions for the existence of equilibria (or fixed points) for a class of correspondences F, defined on a nonempty compact subset M of $\mathbb{R}^n$, with values in $\mathbb{R}^n$, when the domain M is neither assumed to be convex nor smooth. The correspondence F is assumed to be upper semicontinuous with nonempty, convex, compact values, and it said to be "tangent" to M if, for every $x \in M$, $F(x) \cap T_M(x)$ is nonempty (where $T_M(x)$ [resp. $N_M(x)$] denotes Clarke's tangent [resp. normal] cone to M at x).

Our first result states that, if the domain M is epi-Lipschitzian, then Assertion (χ): the set M has at least one connected component with a nonzero Euler characteristic, is necessary and sufficient for the existence of an equilibrium x^* of F on M (i.e., $x^* \in M$ such that $0 \in F(x^*)$), if F is "tangent" to M, and is also necessary and sufficient for the existence of a "generalized equilibrium" x^* of F (i.e., $x^* \in M$ such that $0 \in F(x^*) - N_M(x^*)$), if F is no longer assumed to be "tangent" to M.

Our second result extends the sufficient part of both existence results (equilibria and generalized equilibria) to the larger class of locally connected proximally nondegenerate sets, i.e., such that, for $\varepsilon > 0$ small enough, $M_\varepsilon = \overline{B}(M, \varepsilon)$ is epi-Lipschitzian, and converges normally ([14]) to M in the sense that $\limsup \mathrm{G}(N_{M_\varepsilon}) \subset \mathrm{G}(N_M)$, where $\mathrm{G}(N_M)$ is the graph of the correspondence N_M; this class of sets contains, in particular, epi-Lipschitzian sets, convex sets, and smooth submanifolds.

This allows us to provide a unified existence theory of (generalized) equilibria and fixed points, which covers the classical results of Brouwer and Kakutani in the convex case and the existence part of Poincaré-Hopf's theorem in the smooth case.

Key words. *Fixed point, (generalized) equilibrium, existence, tangential condition, Clarke's tangent and normal cones, normal convergence, epi-Lipschitzian set, proximally nondegenerate set.*

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1 Introduction

The aim of this paper is to provide a sufficient condition for the existence of equilibria, generalized equilibria and fixed points for correspondences F defined on a subset M of $\mathbb{R}^n$ with values in $\mathbb{R}^n$ when the domain M is neither assumed to be convex nor smooth. We shall also show that the sufficient condition is also necessary for the class of epi-Lipschitzian subsets of $\mathbb{R}^n$.

Let us now present the general framework of the paper. We assume that the correspondence F is upper semicontinuous (u.s.c.), i.e., the set $\{x \in M | F(x) \subset V\}$ is open in M for every open set $V \subset \mathbb{R}^n$, and $F(x)$ is nonempty, compact, and convex, for every $x \in M$. Note that, when F is single-valued (i.e., for every $x \in M$, the set $F(x)$ is reduced to a singleton $\{f(x)\}$), these assumptions are equivalent to saying that the map $f : M \to \mathbb{R}^n$ is continuous.

If $M \subset \mathbb{R}^n$ is closed, and $x \in M$, we let $N_M(x)$ [resp. $T_M(x)$] be Clarke's normal [resp. tangent] cone to M at x. First, we consider the class $\mathcal{L}$ of closed epi-Lipschitzian subsets of $\mathbb{R}^n$, a class of sets introduced by Rockafellar [25] in optimization. A closed subset M of $\mathbb{R}^n$ is said to be epi-Lipschitzian if, at every $x \in M$, $N_M(x)$ is pointed, that is, $N_M(x) \cap -N_M(x) = \{0\}$, which is equivalent to saying that M can be locally written as the epigraph of a Lipschitzian function. Then we consider the larger class $\mathcal{P}$ of locally connected and proximally nondegenerate subsets M of $\mathbb{R}^n$, where a closed subset M of $\mathbb{R}^n$ is said to be proximally nondegenerate if, for $\varepsilon > 0$ small enough, $M_\varepsilon = \overline{B}(M, \varepsilon)$ is epi-Lipschitzian, and converges normally ([14]) to M in the sense that $\limsup \mathrm{G}(N_{M_\varepsilon}) \subset \mathrm{G}(N_M)$, where $\mathrm{G}(N_M)$ is the graph of the correspondence N_M. We show that the class $\mathcal{P}$ contains the class $\mathcal{L}$, together with other examples of particular importance for applications (Propositions 2.2 and 2.5). Indeed, the class $\mathcal{P}$ [resp. $\mathcal{L}$] contains both the class of closed convex subsets of $\mathbb{R}^n$ [resp., with nonempty interiors] and the class of closed continuously differentiable submanifolds of $\mathbb{R}^n$ with boundaries or with corners [resp., of full dimension in $\mathbb{R}^n$].

Our second assumption on the set M is a topological one, which involves the Euler characteristic $\chi(M)$ of M. In this paper, we shall follow the geometric approach of Cornet [12] for compact sets in $\mathcal{L}$ and then extend it to compact sets in $\mathcal{P}$ by letting:

$$\chi(M) = \chi(\overline{B}(M, \varepsilon)),$$

which is constant for $\varepsilon > 0$ small enough (see Definition 2.7). At this stage, we only need to know that, if M is compact and belongs to $\mathcal{L}$ [resp., to $\mathcal{P}$], then there is a finite number of connected component M_i of M, each of which is compact and also belongs to $\mathcal{L}$ [resp. $\mathcal{P}$] (see Corollary 3.5 and Proposition 4.2). Our topological assumption, for a compact set M in $\mathcal{P}$ (hence also for a set in $\mathcal{L}$), states:

Assertion (χ) *There is a connected component M_i of M such that $\chi(M_i) \neq 0$,*

and is satisfied if $\chi(M) \neq 0$, since $\chi(M) = \sum_{i \in I} \chi(M_i)$ (but the converse may not be true), and if M is convex (in this case, $\chi(M) = 1$).

In this paper, we show that, for compact sets in $\mathcal{P}$, Assertion (χ) is sufficient for the existence of equilibria and that, for compact sets in $\mathcal{L}$, it is also necessary (Theorem 3.1 and Corollary 3.5). More precisely, for a nonempty, compact set in $\mathcal{P}$, Assertion (χ) implies the following Assertion (EQ), and the converse is true if M is additionally assumed to belong to $\mathcal{L}$.

Assertion (EQ) **[Equilibria]** *For every u.s.c. correspondence F from M to $\mathbb{R}^n$, with nonempty convex compact values, such that:*

$$\forall x \in \operatorname{bd} M,\ F(x) \cap T_M(x) \neq \emptyset,$$

there exists an equilibrium $x^* \in M$ of F, in the sense that:

$$0 \in F(x^*).$$

Assertion (EQ) can be equivalently reformulated in terms of existence of fixed points:

Assertion (FP) **[Fixed point]** *For every u.s.c. correspondence F from M to $\mathbb{R}^n$, with nonempty convex compact values, such that:*

$$\forall x \in \operatorname{bd} M,\ F(x) \cap \Big(\{x\} + T_M(x)\Big) \neq \emptyset,$$

there exists a fixed point $x^* \in M$ of F, in the sense that:

$$x^* \in F(x^*).$$

In the absence of tangential condition on F, we shall consider the existence of equilibria of the ["variational inequality"] correspondence $F - N_M$, also called "generalized equilibria" of F. Formally, for a nonempty, compact set in $\mathcal{P}$, Assertion (χ) implies the following Assertion (GE), and the converse is true if M is additionally assumed to belong to $\mathcal{L}$.

Assertion (GE) **[Generalized Equilibria]** *For every u.s.c. correspondence F from M to $\mathbb{R}^n$, with nonempty, convex, compact values, there exists a "generalized" equilibrium $x^* \in M$ of F in the sense that:*

$$0 \in F(x^*) - N_M(x^*).$$

We now discuss the relations between our results and previous ones in the literature. Our results generalize known results in the finite dimensional setting, both in the convex case and in the smooth case. If M is convex, the implication $[(\chi) \Rightarrow (FP)]$ is the existence theorem of fixed points for inward (or outward) correspondences proved in a long tradition of authors in the more general setting of infinite dimensional spaces (for example, Bergman and Halpern, [3], Cornet [11]), which also generalizes the classical results of Brouwer and Kakutani. If M is smooth (i.e., is a C^2 submanifold with a boundary of $\mathbb{R}^n$, of full dimension), the implication $[(\chi(M) \neq 0) \Rightarrow (EQ)]$, in the single-valued case, is essentially the existence part of Poincaré-Hopf's theorem (for example, see Milnor [22]). The equivalence $[(\chi(M) \neq 0) \Leftrightarrow (EQ)]$ is well known when M is additionally assumed to be connected and the correspondence F to be single-valued (for example, see Hirsch [20]).

In the nonsmooth and nonconvex case, the existence problem of equilibria has been considered by several authors (Cornet [12], Clarke, Ledyaev and Stern [9], Ben-El-Mechaiekh and Kryszewski [1]). In [12], the implications $[(\chi(M) \neq 0) \Rightarrow (EQ) \Rightarrow (GE)]$ are proved if M is nonempty, compact, and epi-Lipschitzian, using degree theory. In [9], it is shown that Assertion (EQ) holds if M is compact, epi-Lipschitzian and homeomorphic to a convex set (a condition which implies that $\chi(M) = 1$) in $\mathbb{R}^n$ and, possibly, in an infinite dimensional setting using techniques from differential equations and fixed-point theorems (Brouwer, Kakutani). Finally, [1] shows the implication $[\chi(M) \neq 0 \Rightarrow (EQ)]$ for the class of compact $\mathcal{L}$-retract sets in an infinite dimensional setting, using Lefschetz fixed-point theorem. The implication $[(\chi) \Rightarrow (EQ)]$ of our result is not comparable to [1] since one can find sets in $\mathcal{P}$ which are not $\mathcal{L}$-retracts (see the appendix).

The existence of generalized equilibria is quite standard in the convex case, in which the generalized equation $0 \in F(x) - N_M(x)$ is usually called variational inequality. In the nonconvex case, Cornet [12] considers the epi-Lipschitzian case and proves the implication $[(\chi(M) \neq 0) \Rightarrow (GE)]$. We extend this result to the larger class $\mathcal{P}$. Finally, it is worth pointing out that our existence result of equilibria is deduced from our existence result of generalized equilibria, which thus provides a unified and more general setting to study the existence problem.

The paper is organized as follows. In Section 2, we recall some definitions and results on the topological degree of correspondences, the Euler characteristic, and the class of sets $\mathcal{L}$ and $\mathcal{P}$ which are considered. In Section 3, we state our main results. We first consider the epi-Lipschitzian case for which we show (Theorem 3.1) that Assertion (χ) is necessary and sufficient for Assertions (EQ) and (GE) to hold. We then extend (Corollary 3.5) to the class $\mathcal{P}$ the sufficient part of our previous result, i.e., $[(\chi) \Rightarrow (EQ)]$ and $[(\chi) \Rightarrow (GE)]$, which is deduced from a more general existence result (Theorem 3.2) on the class $\overline{\mathcal{L}}$ of "normal limits" of epi-Lipschitzian sets. In Section 4, we prove Theorem 3.1 and in Section 5, we prove Theorem 3.2. Finally, in the appendix, we give further properties of the classes $\mathcal{L}$, $\overline{\mathcal{L}}$ and $\mathcal{P}$, and we discuss the link between our definition of the Euler characteristic and the classical (topological) one using the homology groups of M.

2 Preliminaries [2]

2.1 The topological degree of correspondences

A correspondence F from $X \subset \mathbb{R}^m$ to $\mathbb{R}^n$ is a map from X to the set of all the subsets of $\mathbb{R}^n$ and the graph of F is defined by $\mathrm{G}(F) = \{(x,y) \in X \times \mathbb{R}^n | y \in F(x)\}$.[2] The correspondence F is said to be upper semicontinuous (u.s.c.), if the set $\{x \in X | F(x) \subset V\}$ is open in X for every open set $V \subset \mathbb{R}^m$.

Let us consider the set $\mathcal{C}$ of the triples (F, Ω, y), where Ω is a bounded open subset of $\mathbb{R}^n$, F is an u.s.c. correspondence defined on $\mathrm{cl}\,\Omega$, with nonempty compact convex values in $\mathbb{R}^n$, and $y \in \mathbb{R}^n \setminus F(\mathrm{bd}\,\Omega)$. We recall the following definition and axiomatic characterization of the degree in the multivalued case (see Cellina & Lasota [7], Granas [18]):

Theorem 2.1 *There exists one and only one map* $\deg : \mathcal{C} \to \mathbb{Z}$ *satisfying the following properties:*
(i) [**Normalization**] $\deg(\mathrm{id}_\Omega, \Omega, y) = 1$ *for all* $y \in \Omega$ *(where* id_Ω *is the correspondence defined by* $\mathrm{id}_\Omega(x) = \{x\}$*).*
(ii) [**Additivity**] $\deg(F, \Omega, y) = \deg(F, \Omega_1, y) + \deg(F, \Omega_2, y)$ *if* Ω_1 *and* Ω_2 *are disjoint open subsets of* Ω *such that* $y \notin F(\mathrm{cl}\,\Omega \setminus (\Omega_1 \cup \Omega_2))$.
(iii) [**Homotopy invariance**] $\deg(H(t,.), \Omega, y)$ *does not depend on* t*, where* H *is an u.s.c. correspondence from* $[0,1] \times \mathrm{cl}\,\Omega$ *to* $\mathbb{R}^n$*, with nonempty compact convex values, such that* $y \notin H(t,x)$ *for all* $(t,x) \in [0,1] \times \mathrm{bd}\,\Omega$.

Remark. In the following, when there is no ambiguity, we shall identify a continuous map $f : \mathrm{cl}\,\Omega \to \mathbb{R}^n$ with its associated correspondence $\hat{f}$ defined by $\hat{f}(x) = \{f(x)\}$. This allows us to define $\deg(f, \Omega, y)$ as follows:

$$\deg(f, \Omega, y) = \deg(\hat{f}, \Omega, y),$$

which coincides with Brouwer's degree for continuous maps.

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[1] We let $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$ and sgn $x = x/|x|$ if $x \in \mathbb{R} \setminus \{0\}$. If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ belong to $\mathbb{R}^n$, we denote $(x|y) = \sum_{i=1}^n x_i y_i$, the scalar product of $\mathbb{R}^n$, $\|x\| = \sqrt{(x|x)}$, the Euclidean norm; we denote $B(x,r) = \{y \in \mathbb{R}^n | \; \|x-y\| < r\}$, $\overline{B}(x,r) = \{y \in \mathbb{R}^n | \; \|x-y\| \leq r\}$, $S(x,r) = \{y \in \mathbb{R}^n | \; \|x-y\| = r\}$, $S = S(0,1)$ the unit sphere. If $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^n$, and $x \in \mathbb{R}^n$, we denote $d_X(x) = \inf_{y \in X} \|x-y\|$, $X \setminus Y = \{x \in X | x \notin Y\}$ the set-difference of the sets X and Y, $X + Y = \{x + y | x \in X, y \in Y\}$, the sum of the sets X and Y, $B(X,r) = X + B(0,r)$, $\overline{B}(X,r) = X + \overline{B}(0,r)$, cl$X$ or $\overline{X}$, the closure of X, int$X$, the interior of X, bd$X$ =cl$X \setminus$ int$X$, the boundary of X, $X^\circ = \{y \in \mathbb{R}^n | \forall x \in X, \; (y|x) \leq 0\}$, the negative polar cone of X, co$X$, the convex hull of X. If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}^n$, then $\nabla f(x)$ denotes the gradient of f at x. A map $f : X \to \mathbb{R}$ is locally Lipschitzian if, for every $x \in X$, there is $\varepsilon > 0$ such that $\|f(x_1) - f(x_2)\| \leq \|x_1 - x_2\|$ for every x_1 and x_2 in $B(x, \varepsilon)$

[2] If F_1 and F_2 are two correspondences from $X \subset \mathbb{R}^m$ to $\mathbb{R}^n$, the correspondences $F_1 \cap F_2$, coF_1, are defined, respectively, by $(F_1 \cap F_2)(x) = F_1(x) \cap F_2(x)$, $(\mathrm{co}\,F_1)(x) = \mathrm{co}\,F_1(x)$. A map $f : X \to \mathbb{R}^n$ is called a selection of F_1, if $f(x) \in F_1(x)$ for all $x \in X$. If A is a subset of X, we denote $F_1(A) = \cup_{x \in A} F_1(x)$ and we define the restriction of F_1 to A, denoted $F_1|_A$, to be the correspondence from A to $\mathbb{R}^n$ defined by $F_1|_A(x) = F_1(x)$.

¿From the above theorem, one deduces the following properties of the degree (see [7], for example):

Proposition 2.1 *Let* (F_1, Ω, y) *and* (F_2, Ω, y) *be in* $\mathcal{C}$.
(i) **[Existence]** *If* $\deg(F_1, \Omega, y) \neq 0$*, then there exists* $x \in \Omega$ *such that* $y \in F_1(x)$.
(ii) **[Boundary]** *If* $F_1|_{\mathrm{bd}\,\Omega} = F_2|_{\mathrm{bd}\,\Omega}$*, then* $\deg(F_1, \Omega, y) = \deg(F_2, \Omega, y)$.
(iii) **[Selection]** *If, for every* $x \in \mathrm{cl}\,\Omega$*,* $F_1(x) \subset F_2(x)$*, then* $\deg(F_1, \Omega, y) = \deg(F_2, \Omega, y)$.
(iv) **[Excision]** $\deg(F_1, \Omega, y) = \deg(F_1, \Omega_1, y)$*, for every open subset* Ω_1 *of* Ω *such that* $y \notin F_1(\mathrm{cl}\,\Omega \setminus \Omega_1)$.

2.2 Euler characteristic of epi-Lipschitzian sets

We first recall the definition of Clarke's subdifferential of a locally Lipschitzian function. Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a function. If f is differentiable at $x \in U$, we denote $\nabla f(x)$ the gradient of f at x. If f is locally Lipschitzian, its subdifferential $\partial f(x)$ at $x \in U$ is defined by:

$$\partial f(x) = \mathrm{co}\{\lim_{k\to\infty} \nabla f(x_k) | x_k \to x,\ x_k \in \mathrm{Dom}\,(\nabla f)\},$$

where $\mathrm{Dom}\,(\nabla f)$ is the set on which f is differentiable. Recalling, from Rademacher's theorem, that the set $U \setminus \mathrm{Dom}\,(\nabla f)$ has (Lebesgue) measure zero, one deduces that $\partial f(x) \neq \emptyset$. If $f : U \to \mathbb{R}$ is locally Lipschitzian, we recall that the correspondence $x \mapsto \partial f(x)$ is u.s.c., with nonempty convex compact values (see Clarke [8] Proposition 2.1.5 (d)). If f is of class C^1 on a neighborhood of x, then $\partial f(x) = \{\nabla f(x)\}$.

Let M be a nonempty closed subset of $\mathbb{R}^n$, recalling that d_M, the distance function to M, is Lipschitzian (of constant 1), Clarke's normal [resp. tangent] cone to M at x is then defined by:

$$N_M(x) = \mathrm{cl}\,(\cup_{\lambda \geq 0} \lambda \partial d_M(x)) \quad [\text{resp. } T_M(x) = N_M(x)^\circ]$$

We now recall the definition of epi-Lipschitzian sets.

Definition 2.1 *A closed subset* M *of* $\mathbb{R}^n$ *is said to be epi-Lipschitzian if, for every* $x \in \mathrm{bd}\,M$*,* $N_M(x)$ *is pointed, i.e.,* $N_M(x) \cap -N_M(x) = \{0\}$.

Examples of epi-Lipschitzian sets are given in the next proposition. See Clarke [8] for the proof and the appendix for other characterizations, together with the original definition from Rockafellar [25].

Proposition 2.2 *Let* M *be a closed subset* $\mathbb{R}^n$*, it is epi-Lipschitzian in each of the two following cases :*

(i) *M is a convex subset of* $\mathbb{R}^n$*, with a nonempty interior;*
(ii) *M is a submanifold with a boundary, or with corners, of* $\mathbb{R}^n$*, of class* C^p *(*$1 \leq p \leq +\infty$*) and of full dimension in* $\mathbb{R}^n$*;* [3]
(iii) *M is smooth in the sense that, for every* $x \in \mathrm{bd}\,M$*,* $N_M(x)$ *is a closed half-line, i.e., it can be written as* $\{\lambda g | \lambda \in \mathbb{R}_+\}$ *for some* $g \in \mathbb{R}^n$*.* [4]

We now recall the definition of a Gauss correspondence associated with an epi-Lipschitzian subset of $\mathbb{R}^n$. This definition generalizes the notion of the Gauss map of a smooth set (see, for example, Milnor [22]).

Definition 2.2 *Let M be a closed subset of $\mathbb{R}^n$, we call Gauss correspondence of M, every u.s.c. correspondence G_M, with nonempty compact convex values from $\mathbb{R}^n$ to $\overline{B}(0,1)$, such that:*

$$G_M(x) = \mathrm{co}\,(N_M(x) \cap S) \text{ for all } x \in \mathrm{bd}\, M.$$

If the correspondence G_M is single-valued, i.e., there is a continuous map $g_M : \mathbb{R}^n \to \mathbb{R}^n$ such that $G_M(x) = \{g_M(x)\}$, then g_M is called a Gauss map of M.

Note that, in this paper, the Gauss correspondences are defined on the whole space, so the Gauss map to a smooth set is a continuous extension of Milnor's map to the whole space.

The following proposition shows the existence of a Gauss correspondence in the epi-Lipschitzian case.

Proposition 2.3 *Let M be a nonempty closed epi-Lipschitzian [resp. smooth] subset of $\mathbb{R}^n$. Then there is a Gauss correspondence [resp. map] of M.*

Admitting Proposition 2.3, the proof of which is given below, we now recall the geometric definition of the Euler characteristic, given by Cornet [12].

If M is a nonempty compact epi-Lipschitzian subset of $\mathbb{R}^n$, and if G_M is a Gauss correspondence of M, then clearly:

$$\forall x \in \mathrm{bd}\, M,\ 0 \notin G_M(x). \tag{1}$$

Hence, $\deg(G_M, \mathrm{int}\, M, 0)$ is defined and independent of the choice of G_M, from the boundary property of the degree. Hence the following definition has a clear meaning.

Definition 2.3 *Let M be a nonempty compact epi-Lipschitzian subset of $\mathbb{R}^n$, then the Euler characteristic of M is defined as follows:*

$$\chi(M) = \deg(G_M, \mathrm{int}\, M, 0),$$

where G_M is any Gauss correspondence of M.

We shall show (see the appendix) that the above definition of the Euler characteristic coincides with the usual topological definition involving the homology groups of M.

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[3] We recall that M is a submanifold [resp. with a boundary, resp. with corners] of $\mathbb{R}^n$, of class C^p ($1 \leq p \leq +\infty$) and of dimension k, if, for all $x \in M$, there is a neighborhood U of x, such that $M \cap U$ is C^p diffeomorphic to an open subset of R^k [resp. of $R^{k-1} \times \mathbb{R}_+$, resp. of R^k_+].

[4] See Proposition 6.1 in the appendix for equivalent formulations and the link with Assertion (*ii*).

We now show the existence of a Gauss correspondence.

Proof of Proposition 2.3. Let M be a nonempty closed epi-Lipschitzian [resp. smooth] subset of $\mathbb{R}^n$. The existence of a Gauss correspondence [resp. Gauss map] of M is a consequence of the two following propositions. The first proposition shows the existence of a correspondence G_M [resp. a map g_M] defined on $\mathrm{bd}\, M$. The second proposition allows to extend it to the whole space.

Proposition 2.4 *Let M be a closed epi-Lipschitzian subset of $\mathbb{R}^n$, then:*
(a) $M = \mathrm{cl}\,(\mathrm{int}\, M)$, and $\mathrm{bd}\, M = \mathrm{bd}\,(\mathrm{int}\, M)$;
(b) the correspondence $x \mapsto N_M(x)$, from M to $\mathbb{R}^n$, has a closed graph;
(c) the correspondence $x \mapsto \mathrm{co}\,(N_M(x) \cap S)$, from $\mathrm{bd}\, M$ to $\mathbb{R}^n$, is u.s.c., with nonempty compact convex values.

Assertions (a) and (b) of Proposition 2.4 are a consequence of Rockafellar [25]. Assertion (c) is a consequence of Assertion (b).

The next theorem is due to Cellina [6] in the multivalued case and is Dugundji's extension of Tietze's theorem in the single-valued case.

Theorem 2.2 *Let F [resp. f] be an u.s.c. correspondence, with nonempty compact convex values [resp. a continuous map] defined on a closed set $X \subset \mathbb{R}^m$, with values in $\mathbb{R}^n$. Then there is an u.s.c. correspondence $\widehat{F}$, with nonempty compact convex values, [resp. a continuous map $\hat{f}$] defined on $\mathbb{R}^m$ with values in $\mathbb{R}^n$, such that $\widehat{F}|_X = F$ [resp. $\hat{f}|_X = f$] and such that $\widehat{F}(\mathbb{R}^m) \subset \mathrm{co}\, F(X)$ [resp. $\hat{f}(\mathbb{R}^m) \subset \mathrm{co}\, f(X)$].*

2.3 Normal limits of epi-Lipschitzian sets

If (M_k) is a sequence of subsets of $\mathbb{R}^n$, its upper limit, in the sense of Painlevé-Kuratowski, is defined by:

$$\limsup M_k = \{x \in \mathbb{R}^n | \exists (x_k) \subset \mathbb{R}^n, \exists \varphi \in \mathcal{I}, x_k \to x, x_k \in M_{\varphi(k)} \text{ for all } k\},$$

where $\mathcal{I}$ is the set of all increasing maps $\varphi : \mathbb{N} \to \mathbb{N}$.

We now extend the class $\mathcal{L}$, of epi-Lipschitzian sets, by considering the class $\overline{\mathcal{L}}$ of "limits" of epi-Lipschitzian sets, for the "normal convergence", in the sense of [14],[5] the definition of which is given below:

Definition 2.4 *We say that a sequence (M_k) of closed subsets of $\mathbb{R}^n$ converges normally to a closed subset M of $\mathbb{R}^n$ if the following assertion holds:*

$$(nc) \quad \textit{[normal convergence]} \quad \limsup \mathrm{G}(N_{M_k}) \subset \mathrm{G}(N_M).$$

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[5] This definition is more general than the one given in [14], in which it is additionally assumed $M = \limsup M_k = \liminf M_k$, a condition which was important to relate the topological properties of M and M_k. Note that Definition 2.4 implies that $M = \limsup M_k$.

We now define the class $\overline{\mathcal{L}}$ as follows:

Definition 2.5 *We let $\overline{\mathcal{L}}$ be the class of closed subsets M of $\mathbb{R}^n$ such that there exists a sequence (M_k) of closed subsets of $\mathbb{R}^n$ which converges normally to M and such that, for k large enough, M_k is epi-Lipschitzian and $M_k \subset B(M,1)$.*

Remark 1. The class $\overline{\mathcal{L}}$ coincides with the class of "normal limits" of smooth sets as shown in the appendix (Proposition 6.3). However, to show in practical examples that a set belongs to $\overline{\mathcal{L}}$, the above definition is easier to work with. Indeed, the distance function d_M being Lipschitzian, one may consider a sequence of sets of the type $M_\varepsilon = \{x \in \mathbb{R}^n | d_M(x) \leq \varepsilon\}$ which is epi-Lipschitzian whenever $0 \notin \partial d_M(x)$ for every x such that $d_M(x) = \varepsilon$ (see Clarke [8] and also Proposition 6.2).

To give practical examples of classes of sets contained in $\overline{\mathcal{L}}$, we first introduce a definition.

Definition 2.6 *A closed subset M of $\mathbb{R}^n$ is said to be proximally nondegenerate if it satisfies the two following conditions:*

(i) $\forall x \in M$, $\exists \varepsilon > 0$, $\forall x' \in B(x,\varepsilon) \setminus M$, $0 \notin \partial d_M(x')$*;*

(ii) for every sequence (x_k, δ_k) in $(\mathbb{R}^n \setminus M) \times \mathbb{R}^n$, such that $\delta_k \in \partial d_M(x_k)$ for every k, and $(x_k, \delta_k/\|\delta_k\|)$ converges to some $(x,p) \in M \times S$, then $p \in \mathrm{cl}\,(\cup_{\lambda \geq 0} \lambda \partial d_M(x))$.

Remark 2. One easily shows that, in the compact case, the above analytic definition is equivalent to the more geometric one given in the introduction, i.e.:

(i') for $\varepsilon > 0$ small enough, $M_\varepsilon = \overline{B}(M,\varepsilon)$ is epi-Lipschitzian;

(ii') $\limsup \mathrm{G}(N_{M_\varepsilon}) \subset \mathrm{G}(N_M)$.

Note that, if the set M only satisfies Assertion (i), then it may not belong to $\overline{\mathcal{L}}$ (see Remark 4, Section 3).

The following proposition shows that the class $\overline{\mathcal{L}}$ contains the class of proximally nondegenerate subsets of $\mathbb{R}^n$, which itself contains the classes of convex subsets of $\mathbb{R}^n$, of smooth submanifolds of $\mathbb{R}^n$, and of proximally smooth subsets of $\mathbb{R}^n$ (see Clarke, Stern and Wolenski [10], Poliquin and Rockafellar [24]).

Proposition 2.5 *Let M be a closed subset of $\mathbb{R}^n$.*
(a) If M is proximally nondegenerate and compact, then M belongs to $\overline{\mathcal{L}}$. Moreover, in Definition 2.5, one can take the sequence of sets $M_k = \overline{B}(M, 1/k)$ $(k \geq 1)$, which converges normally to M and consists of epi-Lipschitzian sets for k large enough.
(b) The set M is proximally nondegenerate in each of the following cases:

- (*i*) *M is a convex subset of* $\mathbb{R}^n$ *(with possibly an empty interior);*
- (*ii*) *M is a submanifold of* $\mathbb{R}^n$*, of class* C^p *(*$1 \leq p \leq +\infty$*), with or without a boundary, with or without corners (of an arbitrary dimension).*
- (*iii*) *M is epi-Lipschitzian;*
- (*iv*) *M is proximally smooth ([10], [24]) in the sense that, for every* $x \in M$*: there is* $\varepsilon > 0$ *such that* d_M *is differentiable on* $B(x, \varepsilon) \setminus M$*.*

The proof of Proposition 2.5 is given in the appendix.

We now extend the previous definition of the Euler characteristic to the class of proximally nondegenerate sets. We first state a lemma:

Lemma 2.1 *Let* $M \subset \mathbb{R}^n$ *be compact and proximally nondegenerate. Then:*

- *(i)* $\exists \alpha > 0$, $\forall x \in B(M, \alpha) \setminus M$, $0 \notin \partial d_M(x)$;
- *(ii)* *for every* $\varepsilon \in (0, \alpha)$*, the set* $M_\varepsilon = \overline{B}(M, \varepsilon) = \{x \in \mathbb{R}^n | d_M(x) \leq \varepsilon\}$ *is epi-Lipschitzian and:*
$$N_{M_\varepsilon}(x) \subset \cup_{\lambda \geq 0} \lambda \partial d_M(x) \text{ for every } x \in M_\varepsilon;\ ^6$$
- *(iii)* $\chi(M_\varepsilon) = \deg(\partial d_M, \operatorname{int} M_\varepsilon, 0)$ *is constant for* $\varepsilon \in (0, \alpha)$*;*
- *(iv)* *if M is epi-lipschitzian, then* $\chi(M) = \chi(M_\varepsilon)$ *for* $\varepsilon > 0$ *small enough.*

Admitting Lemma 2.1, the proof of which is given below, we can introduce the following definition.

Definition 2.7 *Let* $M \subset \mathbb{R}^n$ *be compact and proximally nondegenerate, we let:*

$$\chi(M) = \chi(\overline{B}(M, \varepsilon)) \text{ for } \varepsilon > 0 \text{ small enough.}$$

In the appendix, we discuss the link between this definition and the classical topological definition of the Euler characteristic involving the homology groups of M.

Proof of Lemma 2.1. *Proof of* (*i*). By a standard argument using the compactness of M.

Proof of (*ii*). Since $0 \notin \partial d_M(x)$ when $d_M(x) = \varepsilon$, it is a consequence of Clarke [8] (see also Proposition 6.2 in the appendix).

Proof of (*iii*). We first show that, for $\varepsilon \in (0, \alpha)$:

$$\chi(M_\varepsilon) = \deg(\partial d_M, \operatorname{int} M_\varepsilon, 0). \tag{2}$$

Indeed, let G_{M_ε} be a Gauss correspondence associated to M_ε and consider the homotopy $H : [0, 1] \times M_\varepsilon \to \mathbb{R}^n$ defined by $H(t, x) = tG_{M_\varepsilon}(x) + (1 - t)\partial d_M(x)$. Then

---

[6] In fact the equality $N_{M_\varepsilon}(x) = \cup_{\lambda \geq 0} \lambda \partial d_M(x)$ holds, but we do not use it in the following.

$0 \notin H(t,x)$ for every $(t,x) \in [0,1] \times \operatorname{bd} M_\varepsilon$, since $G_{M_\varepsilon}(x)$ and $\partial d_M(x)$ are subsets of the pointed cone $\cup_{\lambda \geq 0} \lambda \partial d_M(x) \setminus \{0\}$. Hence from the homotopy invariance of the degree, we get:

$$\chi(M_\varepsilon) = \deg(G_{M_\varepsilon}, \operatorname{int} M_\varepsilon, 0) = \deg(\partial d_M, \operatorname{int} M_\varepsilon, 0).$$

We now show that $\chi(M_\varepsilon)$ is constant. Let $0 < \varepsilon_1 < \varepsilon_2 < \alpha$. Since $0 \notin \partial d_M(x)$ for every $x \in M_{\varepsilon_2} \setminus \operatorname{int} M_{\varepsilon_1}$, from the excision property of the degree, we get:

$$\deg(\partial d_M, \operatorname{int} M_{\varepsilon_2}, 0) = \deg(\partial d_M, \operatorname{int} M_{\varepsilon_1}, 0).$$

Consequently, from (2), we get $\chi(M_{\varepsilon_2}) = \chi(M_{\varepsilon_1})$.

Proof of (iv). Let $0 < \varepsilon < \alpha$, let $\Delta_M(x) = d_M(x) - d_{\mathbb{R}^n \setminus M}(x)$, let G_M be a Gauss correspondence associated to M. Since $\partial \Delta_M(x) \subset G_M(x)$ for every $x \in \operatorname{bd} M$ (Cornet and Czarnecki [13]), from the selection property of the degree, we get:

$$\chi(M) = \deg(G_M, \operatorname{int} M, 0) = \deg(\partial \Delta_M, \operatorname{int} M, 0).$$

Noticing that, if $d_M(x) > 0$, $\Delta_M(x) = d_M(x)$ and $\partial \Delta_M(x) = \partial d_M(x)$, we get $0 \notin \partial d_M(x) = \partial \Delta_M(x)$ for every $x \in M_\varepsilon \setminus M$. We recall that $0 \notin \partial \Delta_M(x)$ for every $x \in \operatorname{bd} M$, since M is epi-Lipschitzian (see [13]). Consequently, $0 \notin \partial \Delta_M(M_\varepsilon \setminus \operatorname{int} M)$, and from the excision property of the degree, we get:

$$\deg(\partial \Delta_M, \operatorname{int} M, 0) = \deg(\partial \Delta_M, \operatorname{int} M_\varepsilon, 0) = \deg(\partial d_M, \operatorname{int} M_\varepsilon, 0).$$

Consequently $\chi(M) = \chi(M_\varepsilon)$.□

3 Statement of the results

We first consider the class $\mathcal{L}$ of epi-Lipschitzian sets and then the larger class $\overline{\mathcal{L}}$ of normal limits of epi-Lipschitzian sets. In the following, we respectively denote (EQ_{sv}), (FP_{sv}), and (GE_{sv}) the single-valued versions of Assertions (EQ), (FP), and (GE), i.e., in which we replace "u.s.c. correspondence F from M to $\mathbb{R}^n$, with nonempty, convex, compact values" by "continuous map $f : M \to \mathbb{R}^n$ ".

3.1 The epi-Lipschitzian case: a necessary and sufficient condition

Our first result state that:
Assertion (χ) *There is a connected component M_i of M such that $\chi(M_i) \neq 0$,*
is necessary and sufficient for the existence of equilibria and generalized equilibria in the epi-Lipschitzian case.

Theorem 3.1 *Let M be a nonempty compact epi-Lipschitzian subset of $\mathbb{R}^n$. Then the seven following assertions (χ), (EQ_{sv}), (EQ), (FP_{sv}), (FP), (GE_{sv}), (GE) are all equivalent.*

The proof of Theorem 3.1 is given in Section 4.

Note that Assertion (χ) may no longer be necessary if M is not assumed to be epi-Lipschitzian (see Remark 2 below). We shall extend the sufficient part of Theorem 3.1 to the class $\overline{\mathcal{L}}$ in the next section.

We now state three previous results in the literature, which can already be deduced from Theorem 3.1. The first two are classical results in the convex case and the finite dimensional setting.

Corollary 3.1 *[Bergman and Halpern] Let $M \subset \mathbb{R}^n$ be convex compact with a nonempty interior. Then Assertion (FP) holds true.*

Proof of Corollary 3.1. If $M \subset \mathbb{R}^n$ is convex compact with a nonempty interior, then $M \in \mathcal{L}$ and $\chi(M) = 1$ (see, for example, Cornet [12]), hence the result follows from Theorem 3.1.□

Corollary 3.2 *[Kakutani] Let $M \subset \mathbb{R}^n$ be convex compact with a nonempty interior, then every u.s.c. correspondence F from M to M, with nonempty convex compact values has a fixed point (i.e., there is $x^* \in M$ such that $x^* \in F(x^*)$.)*

Proof of Corollary 3.2. Corollary 3.2 is an immediate consequence of Corollary 3.1, since every u.s.c. correspondence F from M to M satisfies $F(x) \cap \{x\} + T_M(x) \neq \emptyset$ for all $x \in \operatorname{bd} M$.□

The last result considers epi-Lipschitzian sets M homeomorphic to a convex set, a class of sets considered by Clarke, Ledyaev and Stern [9] for which $\chi(M) = 1$.

Corollary 3.3 *[Clarke, Ledyaev, Stern [9]] Let $M \in \mathcal{L}$ be homeomorphic to a compact convex subset of $\mathbb{R}^n$. Then Assertion (EQ) holds true.*

Proof of Corollary 3.3. Let $M \in \mathcal{L}$ be homeomorphic to a compact convex subset C of $\mathbb{R}^n$. Then C has a nonempty interior, hence C is epi-Lipschitzian and $\chi(C) = 1$ (see [12]). From the appendix, the above definition of the Euler characteristic $\chi(M)$ coincides with the classical topological definition (Proposition 6.4), for which we know (see Dold [16]) that two compact homeomorphic sets have the same Euler characteristic. Consequently $\chi(M) = \chi(C) = 1$ and the result follows from Theorem 3.1.□

Remark 1. Note that the assertion that $\chi(M) \neq 0$ is stronger than Assertion (χ) and that both assertions are equivalent when the set M is connected. The following example shows that the implication $[(EQ_{sv}) \Rightarrow \chi(M) \neq 0]$ may not be true if M is not assumed to be connected. Consider $M = M_1 \cup M_2$, a disjoint union of connected smooth subsets M_1 and M_2 of $\mathbb{R}^n$. Assume that $\chi(M_1) = 1$ and $\chi(M_2) = -1$, then $\chi(M) = 0$, and (EQ) and (GE) clearly hold (apply Theorem 3.1 to one of the connected component M_1 or M_2).

Remark 2. If the set M is not epi-Lipschitzian, the topological assumption (χ) may no longer be necessary for any of the assertions (EQ_{sv}), (EQ), (FP_{sv}), (FP), (GE_{sv}), (GE). In $\mathbb{R}^2$, consider the square:

$$M = \{(x, y) \in \mathbb{R}^2 | \sup\{|x|, |y|\} = 1\}.$$

Note that $\chi(M) = 0$, $N_M(1, 1) = \mathbb{R}^2$ and $T_M(1, 1) = \{0\}$.

3.2 The non epi-Lipschitzian case: a sufficient condition

We now provide an existence result for compact sets in $\overline{\mathcal{L}}$.

Theorem 3.2 *Let* $M \in \overline{\mathcal{L}}$ *be nonempty, compact, i.e., there is a uniformly bounded sequence* (M_k) *of closed epi-Lipschitzian subsets of* $\mathbb{R}^n$ *converging normally to* M*, and assume:*
Assertion $(\overline{\chi})$ $\quad \chi(M_k) \neq 0$ *for* k *large enough.*
Then each of the following assertions (EQ), (FP), (GE) holds.

The proof of Theorem 3.2 is given in Section 5.

Remark 3. Note that the topological assumption $(\overline{\chi})$ in Theorem 3.2 is not formulated in terms of the Euler characteristic of M. It cannot be done with our previous definition which is only given for proximally nondegenerate sets. It cannot also be done with the classical definition, denoted hereafter $\chi_{top}(M)$, involving the homology groups of M (see the appendix); indeed, one may find examples of sets M satisfying the assumptions of Theorem 3.2, for which $\chi_{top}(M)$ is not defined (see Remark 1 in the appendix.).

For the class of proximally nondegenerate sets, we can state the two following results.

Corollary 3.4 *Let* $M \subset \mathbb{R}^n$ *be nonempty, compact and proximally nondegenerate, such that* $\chi(M) \neq 0$*. Then each of the following assertions (EQ), (FP), (GE) holds.*

Proof of Corollary 3.4. From Proposition 2.5, the proximally nondegenerate set M belongs to $\overline{\mathcal{L}}$, and the sequence of sets $M_k = \overline{B}(M, 1/k)$ $(k \geq 1)$ converges normally to M and consists of epi-Lipschitzian sets for k large enough. The assumption $\chi(M) \neq 0$ and Definition 2.7 of the Euler characteristic imply that $0 \neq \chi(M) = \chi(M_k)$ for k large enough. Hence the result follows from Theorem 3.2.$\square$

We now deduce the result stated in the introduction, under the additionnal assumption that M is locally connected, i.e., for every $x \in M$, for every neighborhood V of x, there is a connected neighborhood $W \subset V$ of x (for the relative topology).

Corollary 3.5 *Let $M \subset \mathbb{R}^n$ be nonempty, compact, locally connected, and proximally nondegenerate, and assume:*
Assertion (χ) *There is a connected component M_i of M such that $\chi(M_i) \neq 0$.*
Then each of the following assertions (EQ), (FP), (GE) holds.

Proof of Corollary 3.5. Since M is locally connected, each connected component M_i of M is open in M, hence, since M is compact, there is a finite number of connected components. But each connected component M_i is closed in M, hence is compact since M is compact. Thus, for $\varepsilon > 0$ small enough:

$$B(M, \varepsilon) = \cup_{i=1}^{k} B(M_i, \varepsilon).$$

Then we notice that each M_i is proximally nondegenerate, hence $\chi(M_i)$ is well defined (Definition 2.7). From Assertion (χ) and Corollary 3.4, we get the result.□

Note also that, in view of the additivity property of the degree, we get $\chi(M) = \sum_{i=1}^{k} \chi(M_i)$.

Remark 4. Corollary 3.4 is no longer true if the set M is not assumed to be proximally nondegenerate. Consider the following connected set illustrated in Figure 1:

$$M = \left\{(x,y) \in \mathbb{R}^2 \middle| \begin{array}{ll} & |x| \geq 1/4, \quad (x-(1/4)\operatorname{sgn} x)^2 + (y - \operatorname{sgn} y)^2 \in [1/2, 1], \\ \text{or} & |x| < 1/4, \quad y \in [-2, -3/2] \cup [-1/2, 1/2] \cup [3/2, 2] \end{array} \right\}.$$

Then M is compact, and $\chi_{top}(M) = -1$. Let $f : M \to \mathbb{R}^2$ be the map defined by $f(x,y) = (1 - |y|, (x - \operatorname{sgn} x/4) \operatorname{sgn} y)$ if $|x| > 1/4$ and $f(x,y) = (1-|y|, 0)$ if $|x| \leq 1/4$. The map f is continuous on M, and for all $x \in M$, $f(x) \in T_M(x) \setminus \{0\}$, thus $f(x) \notin N_M(x)$. Hence none of the assertions (*EQ*), (*FP*), (*GE*) holds.

---

[6] In the classical sense, see Appendix.

Figure 1

Note also that $0 \notin \partial d_M(x)$ for every $x \in B(M, 1/2) \setminus M$, and that M does not belong to $\overline{\mathcal{L}}$. Thus Corollary 3.4 cannot be extended to the class of compact subsets M of $\mathbb{R}^n$, which only satisfy:

$$\forall x \in M,\ \exists \varepsilon > 0,\ \forall x' \in B(x, \varepsilon) \setminus M,\ 0 \notin \partial d_M(x').$$

4 Proof of Theorem 3.1

The proof of Theorem 3.1 relies on two types of results on epi-Lipschitzian sets, which will allow us to reduce the proof to the (classical) case of smooth connected domains. These results are stated and proved in the next two sections.

4.1 Reducing to the smooth case

To reduce to the smooth case, we need several results. The following approximation result is taken from Cornet and Czarnecki [14], which also provides a dual formulation in terms of external (decreasing) approximation:

Theorem 4.1 *Let M be a nonempty, compact, epi-Lipschitzian subset of $\mathbb{R}^n$. Then there is a sequence (M_k) of subsets of $\mathbb{R}^n$ satisfying the following properties:*

(i) *for all $k \in \mathbb{N}$, the set M_k is compact and smooth;*

(ii) $\operatorname{int} M = \cup_{k \in \mathbb{N}} M_k$, $M = \operatorname{cl}(\cup_{k \in \mathbb{N}} M_k)$[7], *and for every k, $M_k \subset \operatorname{int} M_{k+1}$;*

(iii) $\limsup \mathrm{G}(N_{M_k}) \subset \mathrm{G}(N_M)$;

(iv) *the sets M and M_k are homeomorphic.*[8]

In this section, a sequence (M_k) satisfying the conclusions of Theorem 4.1 will be called an internal smooth normal approximation of M, or simply a smooth approximation if there is no ambiguity.

The next proposition shows that, if (M_k) is a smooth internal approximation of a nonempty compact epi-Lipschitzian subset M of $\mathbb{R}^n$, then $\chi(M_k) = \chi(M)$ for k large enough.

Proposition 4.1 *Let (M_k) be a smooth internal approximation of a nonempty compact epi-Lipschitzian subset M of $\mathbb{R}^n$, then:*

$$\chi(M_k) = \chi(M), \textit{ for } k \textit{ large enough.}$$

We prepare the proof of Proposition 4.1 with the following lemma:

Lemma 4.1 *Let G_M [resp. g_{M_k}] be a Gauss correspondence [resp. Gauss map] of the set M [resp. M_k]. Then there is a correspondence G from $\mathbb{R}^n$ to $\mathbb{R}^n$ which is u.s.c., with nonempty compact convex values and such that, for k large enough:*
(i) $G_M(x) \subset G(x)$ for every $x \in \operatorname{bd} M$;
(ii) $g_{M_k}(x) \in G(x)$ for every $x \in \operatorname{bd} M_k$;
(iii) $0 \notin G(x)$ for every $x \in M \setminus \operatorname{int} M_k$.

Proof of Lemma 4.1 We let G_M be a Gauss correspondence of M (see Proposition 2.3). For $r > 0$ we define the correspondence G from $\mathbb{R}^n$ to $\mathbb{R}^n$, by:

$$G(x) = \operatorname{co} \overline{B}\Big(G_M(\overline{B}(x,r)), r\Big).$$

Then G has clearly nonempty convex values. Besides, the correspondence G is also u.s.c., with compact values (recalling that, if F is an u.s.c. correspondence with compact values from $\mathbb{R}^n$ to $\mathbb{R}^n$ and if $r > 0$, then the correspondence $\operatorname{co} F$, and the correspondences F_1 and F_2, defined by $F_1(x) = F(\overline{B}(x,r))$ and $F_2(x) = \overline{B}(F(x), r)$ respectively, are also u.s.c., with compact values).

Proof of (i) *and* (ii). Assertion (i) is a direct consequence of the definition of G. We now prove that, for k large enough, Assertion (ii) holds. Suppose it is not true, without any loss of generality, we may assume that there is a sequence (x_k) in $\mathbb{R}^n$ such that, for every k, $x_k \in \operatorname{bd} M_k$ and $g_{M_k}(x_k) \notin G(x_k)$. Since the sequence $(x_k, g_{M_k}(x_k))$ belongs to the compact set $M \times S$, without any loss of generality, we may assume that it converges to some element $(x, p) \in M \times S$. Since (M_k) is a smooth normal approximation of M, we get:

$$(x,p) = \lim(x_k, g_{M_k}(x_k)) \in \limsup \mathrm{G}(N_{M_k}) \subset \mathrm{G}(N_M).$$

---

[7] Note that this property is in fact a consequence of the previous one since $M = \operatorname{cl}(\operatorname{int} M)$, from Proposition 2.4.

[8] The existence of a sequence (M_k) satisfying Assertions (i), (ii), and (iii) is also proved in Cornet-Czarnecki [13]. Then such a sequence satisfies Assertion (iv), for k large enough (see Cornet-Czarnecki [14]).

Hence $p \in N_M(x) \cap S \subset G_M(x)$. For k large enough, $x \in \overline{B}(x_k, r)$ and $g_{M_k}(x_k) \in \overline{B}(p, r)$, hence $g_{M_k}(x_k) \in G_M(\overline{B}(x_k, r)) + \overline{B}(0, r) \subset G(x_k)$, which contradicts that $g_{M_k}(x_k) \notin G(x_k)$. This ends the proof of *(ii)*.□

Proof of (iii). We first show that:

$$\text{for } k \text{ large enough, } M \setminus \operatorname{int} M_k \subset B(\operatorname{bd} M, r).$$

Indeed, if it is not true, without any loss of generality, we may assume that there is a sequence (x_k) such that, for every k, $x_k \in M \setminus \operatorname{int} M_k$ and $d_{\operatorname{bd} M}(x_k) \geq r$. Since M is compact, without any loss of generality, we can suppose that (x_k) converges to some element $x \in M$ such that $d_{\operatorname{bd} M}(x) \geq r$. Hence $x \in \operatorname{int} M = \cup_k \operatorname{int} M_k$, from the definition of the approximation. Since $M_k \subset M_{k+1}$ for every k, there are $\varepsilon > 0$ and an integer k_0 such that $B(x, \varepsilon) \subset M_k$ for all $k \geq k_0$. A contradiction with the fact that $x_k \notin \operatorname{int} M_k$.

The end of the proof of Part *(iii)* consists to choose $r > 0$ as in the following claim (taking $m = n$, $\Omega = \mathbb{R}^n$, $K = \operatorname{bd} M$ and $F = G_M$), recalling that $0 \notin G_M(x)$ if $x \in \operatorname{bd} M$ (Assertion (1), Section 2.2).□

Claim 4.1 *Let Ω be a subset of $\mathbb{R}^n$, let $K \subset \Omega$ be compact, let F be an u.s.c. correspondence from Ω to $\mathbb{R}^m$, with nonempty compact convex values, such that $0 \notin F(x)$ for every $x \in K$. Then there exists $r > 0$ such that:*

$$0 \notin \operatorname{co} \overline{B}\Big(F(\overline{B}(x, r) \cap \Omega), r\Big) \text{ for all } x \in B(K, r).$$

Proof of Claim 4.1. By contraposition. Suppose that there exists a sequence (x^k) in $\mathbb{R}^n$ such that, for every $k \geq 1$, $x^k \in B(K, 1/k)$ and:

$$0 \in \operatorname{co} \overline{B}\Big(F(\overline{B}(x^k, 1/k) \cap \Omega), 1/k\Big).$$

¿From Caratheodory's theorem, there exist $n+1$ elements $(x_i^k, y_i^k, \lambda_i^k)$ $(i = 1, \ldots, m+1)$ in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}_+$ such that:

$$0 = \sum_{i=1}^{m+1} \lambda_i^k y_i^k, \tag{3}$$

with $\sum_{i=1}^{m+1} \lambda_i^k = 1$, $y_i^k \in \overline{B}\Big(F(x_i^k), 1/k\Big)$, $x_i^k \in \overline{B}(x^k, 1/k) \cap \Omega$. Without any loss of generality we can assume that the sequence $(x^k, \lambda_1^k, \ldots, \lambda_{m+1}^k, y_1^k, \ldots, y_{m+1}^k)$ converges to some element $(x^*, \lambda_1^*, \ldots, \lambda_{m+1}^*, y_1^*, \ldots, y_{m+1}^*) \in K \times \Sigma \times \mathbb{R}^{m(m+1)}$, since the sequence belongs to the compact set $\overline{B}(K, 1) \times \Sigma \times (\overline{B}(F(\overline{B}(K, 1) \cap \Omega), 1)))^{m+1}$ where Σ is the unit simplex of $\mathbb{R}^{m+1}$. But, for all $i \in \{1, ..., m+1\}$, the sequence (x_i^k) also converges to x^* (since from above $\|x_i^k - x^k\| \leq 1/k$).

Taking the limit in (3) when $k \to \infty$, we get $0 = \sum_{i=1}^{m+1} \lambda_i^* y_i^*$ and $y_i^* \in F(x^*)$, for all $i \in \{1, ..., m+1\}$, since the correspondence F is u.s.c.. Consequently, $0 \in F(x^*)$ since $F(x^*)$ is convex.□

We can now give the proof of Proposition 4.1.

Proof of Proposition 4.1. Let us first show that:

$$\deg(G, \operatorname{int} M, 0) = \deg(G_M, \operatorname{int} M, 0) = \chi(M). \tag{4}$$

Indeed, we recall that $\deg(G_M, \operatorname{int} M, 0) = \chi(M)$ (Definition 2.3), and we get $\deg(G, \operatorname{int} M, 0) = \deg(G_M, \operatorname{int} M, 0)$ from the selection property of the degree since for all $x \in \operatorname{bd} M$, $G_M(x) \subset G(x)$.

We now show that:

$$\text{for } k \text{ large enough, } \deg(G, \operatorname{int} M_k, 0) = \chi(M_k). \tag{5}$$

Indeed, in view of Lemma 4.1, for k large enough, $0 \notin G(M \setminus M_k)$ and $G_{M_k}(x) = \{g_{M_k}(x)\} \subset G(x)$ for all $x \in \operatorname{bd} M_k$. Since $\chi(M_k) = \deg(G_{M_k}, \operatorname{int} M_k, 0)$ (Definition 2.3), and since $G_{M_k}(x) \subset G(x)$ for all $x \in \operatorname{bd} M_k$, the selection property of the degree implies that $\chi(M_k) = \deg(G, \operatorname{int} M_k, 0)$.

In view of (4) and (5), the proof of Proposition 4.1 is complete if we show that:

$$\deg(G, \operatorname{int} M, 0) = \deg(G, \operatorname{int} M_k, 0).$$

Indeed, it is a consequence of the excision property of the degree, since $0 \notin G(M \setminus \operatorname{int} M_k)$ from Lemma 4.1.□

4.2 Reducing to the connected case

The following result will allow us to reduce the proof to the case where M is connected (in this case, Assertion (χ) reduces to the assertion $\chi(M) \neq 0$):

Proposition 4.2 *Let M be a nonempty, compact, epi-Lipschitzian subset of $\mathbb{R}^n$ and let $(M_i)_{i \in I}$ be the connected components of M.*
(a) Every connected component M_i is open in M (for the relative topology of M), i.e., there is an open subset U_i of $\mathbb{R}^n$ such that $U_i \cap M = M_i$.
(b) For every $i \in I$, M_i is nonempty, closed and:

$$N_{M_i}(x) = N_M(x) \text{ and } T_{M_i}(x) = T_M(x) \text{ for every } x \in M_i,$$

and M_i is epi-Lipschitzian.
(c) If we additionally assume that M is compact then the set I is finite, for every $i \in I$, M_i is compact, and

$$\chi(M) = \sum_{i \in I} \chi(M_i).$$

Proof of Proposition 4.2 *Proof of (a).* We first show that M is locally connected, i.e., for every $x \in M$ and every neighborhood V of x (for the relative topology of M), there is a connected neighborhood $W \subset V$ of x (for the relative topology of M). This is a direct consequence of Assertion (*iii*) of Proposition 6.2 (in the appendix), which is usually given as the definition of epi-Lipschitzian sets. Indeed, let V be an open neighborhood of x, from Proposition 6.2, if V is small enough, then V is homeomorphic to an open subset W of $\operatorname{epi}\varphi$, for some Lipschitzian function $\varphi : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^{n-1}$ is open, and W is again homeomorphic to an open subset X of $\Omega \times \mathbb{R}_+$ [consider $H : W \to X$ defined by $H(u, v) = (u, v - \varphi(u))$]. But clearly X is locally connected, hence V and M are also locally connected.

We now prove Assertion (a). Indeed, let M_i be a connected component of M. Since M is locally connected, for every $x \in M_i$, there is a connected neighborhood U_x of x, hence $U_x \subset M_i$ and M_i is open in M.□

Proof of (b). Every connected component M_i of M is clearly closed (in $\mathbb{R}^n$), since M is closed. From (a), there is an open subset U_i of $\mathbb{R}^n$ such that $U_i \cap M = M_i$, and:

$$N_{M_i}(x) = N_M(x) \text{ and } T_{M_i}(x) = T_M(x) \text{ for every } x \in M_i.$$

Hence, for every $x \in M_i$, $N_{M_i}(x) = N_M(x)$ is pointed, since M is epi-Lipschitzian. Consequently, M_i is also epi-Lipschitzian.□

Proof of (c). From (a), the subsets U_i ($i \in I$) define an open covering of M and the compactness of M implies that the set I is finite. We let G_M be a Gauss correspondence associated to M, from (b) we deduce that G_M is also a Gauss correspondence associated to each M_i for every i. From the definition of the Euler characteristic, and the additivity property of the degree, we get:

$$\chi(M) = \deg(G_M, \operatorname{int} M, 0) = \sum_{i \in I} \deg(G_M, \operatorname{int} M_i, 0) = \sum_{i \in I} \chi(M_i).\square$$

4.3 Assertions (χ) and (EQ_{sv}) are equivalent

In view of Proposition 4.2, without any loss of generality, we may assume in this section that M is connected.

Proof of $[(\chi) \Rightarrow (EQ_{sv})]$**.** Let $\varphi : M \to \mathbb{R}^n$ be a continuous map, such that, for all $x \in \operatorname{bd} M$, $\varphi(x) \in T_M(x)$, and let (M_k) be a smooth approximation of M. From Proposition 4.1, $\chi(M_k) = \chi(M)$ for k large enough, hence $\chi(M_k) \neq 0$ from Assumption (χ). For every integer $k \geq 1$, we let g_{M_k} be a Gauss map of M_k, we let:

$$A_k = \max\Big\{\Big(\varphi(y)|g_{M_k}(y)\Big)|y \in \operatorname{bd} M_k\Big\}, \quad \lambda_k = \max\{A_k, 0\} + 1/k,$$

and we define the map $\psi_k : M_k \to \mathbb{R}^n$ by:

$$\psi_k(x) = \lambda_k g_{M_k}(x) - \varphi(x) \text{ for all } x \in M_k.$$

We now show that:

$$\chi(M_k) = \deg(\psi_k, \operatorname{int} M_k, 0). \tag{6}$$

Indeed, it is a consequence of the invariance of the degree through homotopy, considering the homotopy $H : [0,1] \times M_k \to \mathbb{R}^n$ defined by $H(t,x) = t\psi_k(x) + (1-t)g_{M_k}(x)$; one checks that $0 \notin H(t,x)$ for every $(t,x) \in [0,1] \times \operatorname{bd} M_k$, noticing that $(\psi_k(x)|g_{M_k}(x)) > 0$, for every $x \in \operatorname{bd} M_k$.

The existence property of the degree, (6), and $\chi(M_k) = \chi(M) \neq 0$ imply that there is $x_k \in M_k$ such that:

$$0 = \psi_k(x_k) = \lambda_k g_{M_k}(x_k) - \varphi(x_k). \tag{7}$$

Recalling that, for every k, $\|g_{M_k}\| \leq 1$, $\operatorname{bd} M_k \subset M$, $0 \leq \lambda_k \leq \overline{\lambda}$, where $\overline{\lambda} = \max_M \|\varphi\| + 1$, the sequence $(x_k, \lambda_k, g_{M_k}(x_k))$ belongs to the compact set $M \times [0, \overline{\lambda}] \times \overline{B}(0,1)$. Without any loss of generality, we may assume that it converges to some $(x^*, \lambda^*, p^*) \in M \times [0, \overline{\lambda}] \times \overline{B}(0,1)$. Taking the limit when $k \to \infty$, in (7), since the map φ is continuous, one gets:

$$\|\varphi(x^*)\| = \|\lambda^* p^*\| \leq \lambda^*.$$

Then x^* is an equilibrium of φ (and the proof is complete) if we show that $\lambda^* = 0$. Let us assume on the contrary that $\lambda^* > 0$, then there is $\varepsilon > 0$ and is a sequence (y_k) in $\mathbb{R}^n$, such that, for every k:

$$y_k \in \operatorname{bd} M_k \text{ and } \Big(\varphi(y_k)|g_{M_k}(y_k)\Big) = A_k > \varepsilon.$$

Since the sequence $(y_k, g_{M_k}(y_k))$ belongs to the compact set $M \times \overline{B}(0,1)$, without any loss of generality, we may assume that it converges to some element $(y,g) \in M \times \overline{B}(0,1)$ and $(\varphi(y)|g) \geq \varepsilon$. Since (M_k) is a smooth normal approximation of M, we deduce that:

$$(y,g) = \lim(y_k, g_{M_k}(y_k)) \in \limsup \mathrm{G}(N_{M_k}) \subset \mathrm{G}(N_M).$$

Then $g \in N_M(y)$ and $\varphi(y) \in T_M(y)$ contradict the fact that $(\varphi(y)|g) \geq \varepsilon > 0$.□

Proof of $[(EQ_{sv}) \Rightarrow (\chi)]$**.** We prove Assertion (χ) by contraposition. Assume that $\chi(M) = 0$, and let (M_k) be a smooth approximation of M. From the Approximation theorem (Theorem 4.1), the sets M_k (which are homeomorphic to M) are connected for all k, and $\chi(M_k) = \chi(M)$ from Proposition 4.1, for k large enough. Consequently:

$$\chi(M_k) = 0 \text{ for } k \text{ large enough.} \tag{8}$$

We shall contradict Assertion (EQ_{sv}) by showing the existence of a continuous map $\varphi : M \to S$ such that $\varphi(x) \in T_M(x)$ for every $x \in M$. This will be done in two steps. First by defining φ on $M \setminus M_k$ and then extending it to M.

Lemma 4.2 *For k large enough, there is a continuous map* $\varphi : M \setminus \operatorname{int} M_k \to S$*, such* *that:*

$$\begin{array}{ll} (i) & \forall x \in \operatorname{bd} M,\ \varphi(x) \in \operatorname{int} T_M(x); \\ (ii) & \forall x \in \operatorname{bd} M_k,\ \varphi(x) \in \operatorname{int} T_{M_k}(x). \end{array}$$

Proof of Lemma 4.2. Let G be a correspondence satisfying the conclusion of Lemma 4.1, in particular, for k large enough, the nonempty, convex compact set $G(x)$ does not contain 0. Consequently, by a separation theorem, one deduces that, for all $x \in M \setminus \operatorname{int} M_k$:

$$T(x) = \{v \in \mathbb{R}^n | \forall y \in G(x),\ (v|y) < 0\} \neq \emptyset.$$

Since the correspondence G is u.s.c., one easily shows that, for all $y \in \mathbb{R}^n$, the set $T^{-1}(\{y\}) = \{x \in \mathbb{R}^n | y \in T(x)\}$ is open and, for every $x \in M \setminus \operatorname{int} M_k$, the set $T(x)$ is convex. Consequently, from the following claim,[9] there exists a continuous selection ψ of the correspondence T. The map $\varphi : M \setminus \operatorname{int} M_k \to S$, defined by $\varphi(x) = \psi(x)/\|\psi(x)\|$, clearly satisfies the conclusions of the lemma, from Lemma 4.1 (i) and (ii).□

Claim 4.2 *Let T be a correspondence defined on a nonempty compact subset M of $\mathbb{R}^n$, with nonempty convex values in $\mathbb{R}^m$, and such that, for all $y \in \mathbb{R}^m$, the set $\{x \in \mathbb{R}^n | y \in T(x)\}$ is open in M, for its relative topology. Then there is a continuous selection of T.*

Proof of Claim 4.2. Noticing that $M \subset \cup_{y \in R^n} \{x \in M | y \in F(x)\}$, since M is compact, there exists a finite subset $\{y_1, \ldots, y_p\} \subset \mathbb{R}^n$ such that $M \subset \cup_{i=1}^p \{x \in M | y_i \in F(x)\}$. Let $\alpha_i : M \to \mathbb{R}_+$ $(i = 1, \ldots, p)$ be a continuous partition of unity subordinated to the open covering $\{x \in M | y_i \in F(x)\}$ of M. We define $\varphi : M \to \mathbb{R}^n$ by $\varphi(x) = \sum_{i=1}^p \alpha_i(x) y_i$ and we now show that φ is a continuous selection of F. Indeed, let $x \in M$; if $\alpha_i(x) > 0$ then $y_i \in F(x)$ and the convexity of $F(x)$ implies that $\varphi(x) = \sum_i \alpha_i(x) y_i \in F(x)$. □

We end the proof of the implication $[(EQ_{sv}) \Rightarrow (\chi)]$ as follows. Let $\varphi_1 : M \setminus \operatorname{int} M_k \to S$ be a map given by Lemma 4.2, let $\psi : M_k \to \overline{B}(0,1)$ be a continuous extension to M_k of $\varphi_1|_{\operatorname{bd} M_k}$ (Theorem 2.2). Then we show that:

$$0 = \chi(M_k) = \deg(-\psi, \operatorname{int} M_k, 0). \tag{9}$$

Indeed, we recall that $\chi(M_k) = 0$, and we let $g_{M_k} : \mathbb{R}^n \to \overline{B}(0,1)$ be a Gauss map associated to M_k, so $\chi(M_k) = \deg(g_{M_k}, \operatorname{int} M_k, 0)$. The equality $\deg(g_{M_k}, \operatorname{int} M_k, 0) = \deg(-\psi, \operatorname{int} M_k, 0)$ is a consequence of the homotopy invariance of the degree, considering the homotopy $H : [0,1] \times M_k \to \mathbb{R}^n$ defined by $H(t,x) = t g_{M_k}(x) + (1-t)(-\psi(x))$ (notice that $H(t,x) \neq 0$ on $[0,1] \times \operatorname{bd} M_k$, since $-g_{M_k}(x)$ and $\psi(x)$ belong to $\operatorname{int} T_{M_k}(x)$,

---

[9] At this stage we can deduce it from Michael's theorem (see Michael [21]) since T has nonempty, convex, values and is clearly lower semicontinuous. However we are here in a simple case which allows to give a direct (and standard) proof of the existence of a selection.

a convex set which does not contain 0).

From (9) (see, for example Hirsch [20], Theorem 1.8 or Lemma 2.9), the map $\psi|_{\mathrm{bd}\, M_k}$ can be extended on M_k to a continuous map $\varphi_2 : M_k \to S$. We now define the continuous map $\varphi : M \to S$ by:

$$\begin{aligned} \varphi(x) = \varphi_1(x) &\quad \text{if } x \in M \setminus M_k; \\ \varphi(x) = \varphi_2(x) &\quad \text{if } x \in \operatorname{int} M_k, \end{aligned}$$

which clearly satisfies that $\varphi(x) \in T_M(x)$ for every $x \in M$.□

4.4 Assertions (EQ_{sv}), (EQ), (GE_{sv}), (GE), (FP_{sv}), (FP) are all equivalent

We recall that the set M is assumed to be nonempty, compact and epi-Lipschitzian. We shall prove the implications $[(EQ_{sv}) \Rightarrow (EQ)]$, $[(EQ) \Rightarrow (GE)]$, and $[(GE_{sv}) \Rightarrow (EQ_{sv})]$, which proves that the first four assertions are equivalent ($[(EQ_{sv}) \Rightarrow (EQ) \Rightarrow (GE) \Rightarrow (GE_{sv}) \Rightarrow (EQ_{sv})]$). The remaining equivalences $[(EQ_{sv}) \Leftrightarrow (FP_{sv})]$ and $[(EQ) \Leftrightarrow (FP)]$ are immediate. .

Proof of $[(EQ_{sv}) \Rightarrow (EQ)]$**.** By contraposition. Let F be an u.s.c. correspondence from M to $\mathbb{R}^n$, with nonempty, convex, compact values, such that, for all $x \in \mathrm{bd}\, M$, $F(x) \cap T_M(x) \neq \emptyset$. We let, for $\varepsilon > 0$ and $x \in M$:

$$\begin{aligned} F_\varepsilon(x) &= \operatorname{co} B\Big(F(B(x,\varepsilon) \cap M), \varepsilon\Big); \\ \Phi_\varepsilon(x) &= F_\varepsilon(x) \cap \operatorname{int} T_M(x). \end{aligned}$$

We claim that Φ_ε admits a continuous selection. Formally :

Claim 4.3 *(a) The set* $\{x \in M | y \in \Phi_\varepsilon(x)\}$ *is open in* M *(for its relative topology), for every* $y \in \mathbb{R}^n$*;*
(b) the correspondence Φ_ε*, from* M *to* $\mathbb{R}^n$*, has nonempty convex values;*
(c) there exists a continuous map $\varphi_\varepsilon : M \to \mathbb{R}^n$ *such that, for all* $x \in M$*,* $\varphi_\varepsilon(x) \in \Phi_\varepsilon(x) = F_\varepsilon(x) \cap \operatorname{int} T_M(x)$.

Admitting the claim (the proof of which is given below), we then proceed as follows. Let φ_ε be given by the above claim, then it clearly satisfies Assertion (EQ_{sv}). Consequently, there is $x_\varepsilon \in M$ such that:

$$0 = \varphi_\varepsilon(x_\varepsilon) \in \Phi_\varepsilon(x_\varepsilon) \subset \operatorname{co} \overline{B}\Big(F(\overline{B}(x,\varepsilon) \cap M), \varepsilon\Big).$$

Consequently, from Claim 4.1 (taking $m = n$, $\Omega = M$, $K = M$), we get that $0 \in F(x^*)$ for some $x^* \in M$.□

Proof of Claim 4.3. *Proof of* (a). Let $(\overline{x}, y)\ inM \times \mathbb{R}^n$ such that $y \in \Phi_\varepsilon(\overline{x}) = F_\varepsilon(\overline{x}) \cap \operatorname{int} T_M(\overline{x})$. Since $y \in F_\varepsilon(\overline{x})$, from Carathéodory's theorem, there are $n+1$ elements (x^i, y^i, λ^i) in $\mathbb{R}^n \times \mathbb{R}^n \times [0,1]$ such that $y = \sum_{i=1}^{n+1} \lambda^i y^i$, $\sum_{i=1}^{n+1} \lambda^i = 1$, for every $i \in \{1, ..., n+1\}$, $y^i \in B(F(x^i), \varepsilon)$, and $x^i \in B(\overline{x}, \varepsilon) \cap M$.

We let $r = \varepsilon - \max\{\|\overline{x} - x^i\| | i \in \{1, ..., n+1\}\}$. Take $x \in B(\overline{x}, r) \cap M$, then $x^i \in B(x, \varepsilon)$ for every i and $y \in F_\varepsilon(x)$, hence the set:

$$\{x \in M | y \in F_\varepsilon(x)\} \text{ is open in } M \text{ (for its relative topology).}$$

Thus the proof of Part (a) will be complete if:

$$\{x \in M | y \in \text{int} T_M(x)\} \text{ is open in } M \text{ (for its relative topology),}$$

which is indeed a consequence of Rockafellar [25] [Theorem 2, page 147].

Proof of (b). The correspondence Φ_ε has clearly convex values. It has nonempty values since, for every $x \in M$, $F(x) \cap T_M(x) \neq \emptyset$, hence:

$$\emptyset \neq B(F(x), \varepsilon) \cap \text{int}\, T_M(x) \subset \Phi_\varepsilon(x).$$

Proof of (c). In view of Parts (a) and (b), it is a clear consequence of Claim 4.2, or Michael's selection theorem.□

Proof of $[(EQ) \Rightarrow (GE)]$**.** For the sake of completeness, we recall the proof of Cornet [12] (Corollary 4.4). Let F be an u.s.c. correspondence from M to $\mathbb{R}^n$, with nonempty convex compact values, then F is bounded and there exists $k > 0$ such that $F(x) \subset \overline{B}(0, k)$ for every $x \in M$. We define the correspondence F from M to $\mathbb{R}^n$ by:

$$\Phi(x) = F(x) - N_M(x) \cap \overline{B}(0, k).$$

The correspondence Φ is clearly u.s.c. with nonempty convex compact values and we now show that:

$$\text{for every } x \in \text{bd}\, M, \quad \Phi(x) \cap T_M(x) \neq \emptyset.$$

Indeed, let $x \in \text{bd}\, M$ and let $y \in F(x)$. Since $N_M(x)$ is a closed convex cone, with polar $T_M(x)$ we recall that there exist a unique element $y_N \in N_M(x)$ and a unique element $y_T \in T_M(x)$ such that $y = y_N + y_T$ and $(y_N | y_T) = 0$. Consequently, the element $y_T = y - y_N$ belongs to $\Phi(x)$ since $\|y_N\|^2 \leq \|y\|^2 \leq k$. Consequently, from Assertion (EQ), there exists $x^* \in M$ such that $0 \in \Phi(x^*)$ and x^* is clearly a generalized equilibrium of F.□

Proof of $[(GE_{sv}) \Rightarrow (EQ_{sv})]$**.** Let $f : M \to \mathbb{R}^n$ be a continuous map such that $f(x) \in T_M(x)$ for every $x \in M$. From (GE_{sv}), there exists $x^* \in M$ such that $f(x^*) \in N_M(x^*)$. Consequently, $f(x^*) \in N_M(x^*) \cap T_M(x^*) = \{0\}$, and x^* is clearly an equilibrium of f.□

5 Proof of Theorem 3.2

Let $M \in \overline{\mathcal{L}}$ be nonempty and compact, and let (M_k) be a sequence of closed epi-Lipschitzian subsets of $\mathbb{R}^n$ converging normally to M, such that $M_k \subset B(M, 1)$ for k large enough.

5.1 Proof of $[(\overline{\chi}) \Rightarrow (GE)]$

Let F be an u.s.c. correspondence from M to $\mathbb{R}^n$, with nonempty convex compact values. Since the set M is compact, the correspondence F is bounded, i.e., there is $r > 0$ such that $F(x) \subset \overline{B}(0, r)$ for every $x \in M$. From Theorem 2.2, we can extend the correspondence F to the whole space, i.e., there is an u.s.c. correspondence $\widehat{F}$ from $\mathbb{R}^n$ to $\overline{B}(0, r)$, with nonempty convex compact values, such that $\widehat{F}(x) = F(x)$ for every $x \in M$. Since, for k large enough, M_k is nonempty, epi-Lipschitzian, compact, and since, from Assertion $(\overline{\chi})$, $\chi(M_k) \neq 0$, from Theorem 3.1, there is $x_k \in M_k$ such that $0 \in \widehat{F}(x_k) - N_{M_k}(x_k)$, i.e., there is $p_k \in \widehat{F}(x_k) \cap N_{M_k}(x_k)$. Since the sequence (x_k, p_k) belongs to the bounded set $\overline{B}(M, 1) \times \overline{B}(0, r)$, without any loss of generality, we may assume that it converges to some element $(x^*, p^*) \in \overline{B}(M, 1) \times \overline{B}(0, r)$. Since (M_k) converges normally to M, we get:

$$(x^*, p^*) = \lim(x_k, p_k) \in \limsup \mathrm{G}(N_{M_k}) \subset \mathrm{G}(N_M).$$

Hence $x^* \in M$ and $p^* \in N_M(x^*)$. Since the correspondence $\widehat{F}$ is u.s.c., we get $p^* \in \widehat{F}(x^*) = F(x^*)$ (recalling that $x^* \in M$). Consequently $0 = p^* - p^* \in F(x^*) - N_M(x^*)$, i.e., x^* is a generalized equilibrium of F.□

5.2 Proof of $[(\overline{\chi}) \Rightarrow (EQ)]$

Let F be an u.s.c. correspondence from M to $\mathbb{R}^n$, with nonempty convex compact values, such that $F(x) \cap T_M(x) \neq \emptyset$ for every $x \in M$. We prepare the proof with a lemma.

Lemma 5.1 *For every $\varepsilon > 0$, for k large enough, there exists a continuous map $f_{\varepsilon,k} : M_k \to \mathbb{R}^n$ satisfying the two following conditions:*

$$\begin{array}{ll} (i) & \forall x \in M_k,\ \forall p \in N_{M_k}(x) \cap S,\ (f_{\varepsilon,k}(x)|p) < \varepsilon; \\ (ii) & \mathrm{G}(f_{\varepsilon,k}) \subset B(\mathrm{G}(F), \varepsilon). \end{array}$$

Admitting Lemma 5.1, we prove the implication $[(\overline{\chi}) \Rightarrow (EQ)]$. Let $\varepsilon \in (0, 1]$ be fixed and let, for k large enough, $f_{\varepsilon,k}$ be given by Lemma 5.1. From Theorem 3.1, since M_k is nonempty, epi-Lipschitzian, compact, and $\chi(M_k) \neq 0$, there is $x_{\varepsilon,k} \in M_k$ such that:

$$f_{\varepsilon,k}(x_{\varepsilon,k}) \in N_{M_k}(x_{\varepsilon,k}). \tag{10}$$

Since from Lemma 5.1, Assertion (ii), for k large enough:

$$(x_{\varepsilon,k}\ , f_{\varepsilon,k}(x_{\varepsilon,k})) \in \mathrm{G}(f_{\varepsilon,k}) \subset B(\mathrm{G}(F), \varepsilon),$$

the sequence $(x_{\varepsilon,k}\ , f_{\varepsilon,k}(x_{\varepsilon,k}))_k$ is bounded. Without any loss of generality, we may assume that, when $k \to \infty$, it converges to some element $(x_\varepsilon, p_\varepsilon) \in \overline{B}(\mathrm{G}(F), \varepsilon)$. We now show that:

$$\|f_{\varepsilon,k}(x_{\varepsilon,k})\| \leq \varepsilon, \text{ for } k \text{ large enough.} \tag{11}$$

Indeed, it is immediate if $f_{\varepsilon,k}(x_{\varepsilon,k}) = 0$. If $f_{\varepsilon,k}(x_{\varepsilon,k}) \neq 0$, from (10) and from Lemma 5.1, Assertion (i), we get the result. Taking the limit in (11) when $k \to \infty$, we get $\|p_\varepsilon\| \leq \varepsilon$. Without any loss of generality, we may now assume that $(x_\varepsilon, p_\varepsilon)_\varepsilon$ converges to some element $(x^*, p^*) \in \mathbb{R}^n \times \mathbb{R}^n$ when $\varepsilon \to 0$. ¿From above, since the correspondence F is u.s.c., we deduce that $(x^*, p^*) \in \mathrm{G}(F)$ and that $p^* = 0$. Hence x^* is an equilibrium of F.□.

Proof of Lemma 5.1. For a given $\varepsilon > 0$, we let, for $x \in M_k$:

$$\begin{array}{rcl} F_\varepsilon(x) & = & \mathrm{co}\, B\Big(F(B(x,\varepsilon) \cap M), \varepsilon\Big), \\ T_{\varepsilon,k}(x) & = & \{y \in \mathbb{R}^n | \forall p \in N_{M_k}(x) \cap S,\ (y|p) < \varepsilon\}, \\ \Phi_{\varepsilon,k}(x) & = & F_\varepsilon(x) \cap T_{\varepsilon,k}(x). \end{array}$$

Thus Lemma 5.1 is equivalent to saying that, for k large enough, $\Phi_{\varepsilon,k}$ admits a continuous selection $f_{\varepsilon,k} : M_k \to \mathbb{R}^n$ satisfying $\mathrm{G}(f_{\varepsilon,k}) \subset B(\mathrm{G}(F), \varepsilon)$. This is a consequence of the following claim:

Claim 5.1 *For k large enough, $M_k \subset B(M, \varepsilon)$ and:*
(a) the set $\{x \in M_k | y \in \Phi_{\varepsilon,k}(x)\}$ is open in M_k (for its relative topology), for every $y \in \mathbb{R}^n$;
(b) the correspondence $\Phi_{\varepsilon,k}$, from M_k to $\mathbb{R}^n$, has nonempty, convex values;
(c) there exists a continuous map $f_{\varepsilon,k} : M_k \to \mathbb{R}^n$ such that, for every $x \in M_k$, $f_{\varepsilon,k}(x) \in \Phi_{\varepsilon,k}(x)$;
(d) $\mathrm{G}(f_{\varepsilon,k}) \subset B(\mathrm{G}(F), \varepsilon)$.

Proof of Claim 5.1. The assertion that, for k large enough, $M_k \subset B(M, \varepsilon)$, is a direct consequence of the facts that $M = \limsup M_k$ and that the sequence (M_k) is uniformly bounded.

Proof of (a). Let $y \in \mathbb{R}^n$, by a similar argument as the one in the proof of Claim 4.3, Part (a), the set:

$$\{x \in M | y \in F_\varepsilon(x)\} \text{ is open in } M \text{ (for its relative topology).}$$

We end the proof of Part (a) by showing that the set:

$$\{x \in M | y \in T_{\varepsilon,k}(x)\} \text{ is open in } M \text{ (for its relative topology).}$$

Indeed, let $x \in M_k$ such that $y \in T_{\varepsilon,k}(x)$, we have:

$$N_{M_k}(x) \cap S \subset \{p \in \mathbb{R}^n | (y|p) < \varepsilon\}.$$

Since the set M_k is epi-Lipschitzian, from Proposition 2.4, the correspondence $N_{M_k} \cap S$, from M_k to $\mathbb{R}^n$, is u.s.c. at x. Then there is $r > 0$, such that:

$$\forall x' \in B(x,r) \cap M_k,\ N_{M_k}(x') \cap S \subset \{p \in \mathbb{R}^n | (y|p) < \varepsilon\},$$

hence $y \in T_{\varepsilon,k}(B(x,r) \cap M_k)$.□

Proof of (b). The correspondence $\Phi_{\varepsilon,k}$ has clearly convex values. We now show that, for k large enough, it has also nonempty values. Assume that it is not true. Then, without any loss of generality, we may assume that there is a sequence (x_k) in $\mathbb{R}^n$ such that:

$$\forall k,\ x_k \in M_k \text{ and } \emptyset = \Phi_{\varepsilon,k}(x_k) = F_\varepsilon(x_k) \cap T_{\varepsilon,k}(x_k). \tag{12}$$

Since the sequence (x_k) belongs to the bounded set $B(M,1)$, without any loss of generality, we may assume that it converges to some element $x \in \mathbb{R}^n$. Since $M = \limsup M_k$ we deduce that $x \in M$. Let $y \in F(x) \cap T_M(x)$, which is nonempty by assumption, then $y \in F(x) \subset F_\varepsilon(x)$. From Part ($a$), for k large enough, $y \in F_\varepsilon(x_k)$, hence, from (12), $y \notin T_{\varepsilon,k}(x_k)$. From the definition of the correspondence $T_{\varepsilon,k}$, we deduce that there is $p_k \in N_{M_k}(x_k) \cap S$ such that $(y|p_k) \geq \varepsilon$. Again, without any loss of generality, we may assume that (p_k) converges to some element $p \in S$. Since the sequence (M_k) converges normally to M:

$$(x,p) = \lim(x_k,p_k) \in \limsup \mathrm{G}(N_{M_k}) \subset \mathrm{G}(N_M),$$

hence $p \in N_M(x)$. Since, for every k, $(y|p_k) \geq \varepsilon$, at the limit, when $k \to \infty$, we get $(y|p) \geq \varepsilon$, which contradicts the assumption that $y \in T_M(x)$.□

Proof of (c). In view of Parts (a) and (b), it is a clear consequence of Claim 4.2, or Michael's selection theorem..□

Proof of (d). The proof is a classical argument (see Cellina [6]) which we give below for the sake of completeness. In view of Part (c) we only need to show that:

$$\mathrm{G}(F_{\varepsilon'}) \subset B(\mathrm{G}(F), \varepsilon) \text{ for } \varepsilon' > 0 \text{ small enough.}$$

Indeed, since the correspondence F is u.s.c., for every $x \in M$, there is $\varepsilon_x \in (0, \varepsilon]$ such that $F(B(x,\varepsilon_x) \cap M) \subset B(F(x), \varepsilon/2)$. Since the set M is compact, there is a finite number of elements $x_1, \ldots, x_N$ in M such that $M \subset \cup_{i\in\{1,\ldots,N\}} B(x_i, \varepsilon_{x_i}/2)$. Let $\varepsilon' = \min\{\varepsilon_{x_i}/2 | i \in \{1,\ldots,N\}\}$. Then, for every $x \in M$, there is $i \in \{1,\ldots,N\}$ such that $x \in B(x_i, \varepsilon_{x_i}/2)$, and one has:

$$F(B(x,\varepsilon') \cap M) \subset F(B(x_i, \varepsilon_{x_i}) \cap M) \subset B(F(x_i), \varepsilon/2).$$

Noticing that the set $B(F(x_i), \varepsilon/2)$ is convex, we deduce that:

$$F_{\varepsilon'}(x) = \mathrm{co}\, B(F(B(x,\varepsilon') \cap M), \varepsilon') \subset B(F(x_i), \varepsilon).$$

But $F(x_i) \subset F(B(x,\varepsilon) \cap M)$, hence:

$$F_{\varepsilon'}(x) \subset B(F(B(x,\varepsilon) \cap M), \varepsilon).$$

This implies that $\mathrm{G}(F_{\varepsilon'}) \subset B(\mathrm{G}(F), \varepsilon)$.□

6 Appendix

6.1 More on the classes $\mathcal{L}$ and $\overline{\mathcal{L}}$

We first recall a proposition (see, for example, Cornet and Czarnecki [13]) which gives equivalent definitions of "smooth" sets.

Proposition 6.1 *Let M be a closed subset of $\mathbb{R}^n$. Then the three following assertions are equivalent:*

(i) *M is smooth (i.e., $N_M(x)$ is a closed half-line for every $x \in \mathrm{bd}\, M$);*

(ii) *there is a C^1 function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla f(x) \neq 0$ if $f(x) = 0$ and $M = \{x \in \mathbb{R}^n | f(x) \leq 0\}$;*

(iii) *for all $x \in M$, there are an invertible linear map $A : \mathbb{R}^n \to \mathbb{R}^{n-1} \times \mathbb{R}$, a neighborhood U of x, and a function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, which is C^1 on a neighborhood of the $\mathbb{R}^{n-1}$ component of $A(x)$, such that:*

$$M \cap U = U \cap A^{-1}(\mathrm{epi}\, \varphi),$$

where $\mathrm{epi}\, \varphi = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} | y \geq \varphi(x)\}$*;*

(iv) *M is a closed submanifold with a boundary of $\mathbb{R}^n$, of class C^1 and of full dimension in $\mathbb{R}^n$.*

In the epi-Lipschitzian case, the following proposition gives a characterization which is the analogous of the above proposition.

Proposition 6.2 *Let M be a closed subset of $\mathbb{R}^n$. Then the three following assertions are equivalent:*

(i) *M is epi-Lipschitzian (i.e., $N_M(x) \cap -N_M(x) = \{0\}$ for every $x \in \mathrm{bd}\, M$);*

(ii) *there is a locally Lipschitzian function $f : \mathbb{R}^n \to \mathbb{R}$ such that $0 \notin \partial f(x)$ if $f(x) = 0$ and $M = \{x \in \mathbb{R}^n | f(x) \leq 0\}$;*

(iii) *(Rockafellar [25]) for all $x \in M$, there are an invertible linear map $A : \mathbb{R}^n \to \mathbb{R}^{n-1} \times \mathbb{R}$, a neighborhood U of x, and a function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, which is Lipschitzian on a neighborhood of the $\mathbb{R}^{n-1}$ component of $A(x)$, such that:*

$$M \cap U = U \cap A^{-1}(\mathrm{epi}\, \varphi),$$

where $\mathrm{epi}\, \varphi = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} | y \geq \varphi(x)\}$.

We point out that there is no analogous of Assertion (iv) of Proposition 6.1 in the epi-Lipschitzian case. Indeed, every closed epi-Lipschitzian subset of $\mathbb{R}^n$ is a Lipschitzian submanifold of $\mathbb{R}^n$, with a boundary and of full dimension, but the converse is not true in general (see, for example, Benoist and Czarnecki [2]).

Proof of Proposition 6.2. The equivalence [(i) $\Rightarrow$ (iii)] is due to Rockafellar [25]. For the implication [(ii) $\Rightarrow$ (i)], see Clarke [8], and for the converse [(i) $\Rightarrow$ (ii)], when $M \neq \emptyset$ and $M \neq \mathbb{R}^n$, consider the function $\Delta_M(x) = d_M(x) - d_{\mathbb{R}^n \setminus M}(x)$, see Cornet and Czarnecki [13].□

The next proposition points out that the class $\overline{\mathcal{L}}$ coincides with the class of "normal limits" of smooth sets. Formally:

Proposition 6.3 *Let M be a closed subset of $\mathbb{R}^n$. Then the two following assertions are equivalent:*

- (i) *$M \in \overline{\mathcal{L}}$;*
- (ii) *there is a sequence (M_k) of closed smooth subsets of $\mathbb{R}^n$ which converges normally to M and such that $M_k \subset B(M,1)$ for k large enough.*

Proof of Proposition 6.3. The implication $[(ii) \Rightarrow (i)]$ is clear. To prove the converse $[(i) \Rightarrow (ii)]$, let (M_k) be a sequence of closed epi-Lipschitzian subsets of $\mathbb{R}^n$ which converges normally to M and such that $M_k \subset B(M,1)$ for k large enough. We then approximate "normally" each set M_k by a sequence of smooth sets as in Theorem 4.1 and use a diagonal argument to get a smooth normal approximation of the set M.□

We now discuss the link between the classes of sets considered in this paper and the class of $\mathcal{L}$-retract, introduced by Ben-El-Mechaiekh and Kryszewski [1] (see also Plaskacz [23] for a stronger notion) in the more general setting of a metric space. A closed set $M \subset \mathbb{R}^n$ is said to be an $\mathcal{L}$-retract, if there are an open neighborhood U of M, a continuous map $r : U \to M$, and $k \geq 0$, such that, :

$$\|x - r(x)\| \leq k d_M(x), \text{for every } x \in U.$$

Note that this implies that r is a retraction, i.e., for every $x \in M$, $r(x) = x$.

The class of $\mathcal{L}$-retracts contains the class of compact epi-Lipschitzian subsets of $\mathbb{R}^n$ (see [1]) and the class of compact proximally smooth sets. However, it does not contain the class of compact proximally nondegenerate sets. In $\mathbb{R}^2$, consider:

$$M = S\Big((1,0),1\Big) \cup S\Big((-1,0),1\Big) \cup [-1,1] \times \{0\}.$$

We end this subsection by giving the proof of Proposition 2.5.

Proof of Proposition 2.5. *Proof of* (a). For every integer $k \geq 1$, we let:

$$M_k = \{x \in \mathbb{R}^n | d_M(x) \leq 1/k\}.$$

Then clearly $M_k \subset B(M,1)$ for $k \geq 2$. Since M is compact and proximally nondegenerate, for Lemma 2.1, for k large enough, M_k is epi-Lipschitzian and

$$N_{M_k}(x) \subset \cup_{\mu \geq 0} \mu \partial d_M(x) \text{ for every } x \in M_k. \tag{13}$$

We now show that the sequence (M_k) converges normally to M. Indeed, let $(x,p) \in \limsup \mathrm{G}(N_{M_k})$. Without any loss of generality, we may assume that there is a sequence (x_k, p_k) in $\mathbb{R}^n$ which converges to (x, p) such that, for all k, $x_k \in M_k$ and $p_k \in N_{M_k}(x_k)$. Clearly $x \in M$ since $d_M(x_k) \leq 1/k$. From (13), there are $\mu_k \geq 0$ and $\delta_k \in \partial d_M(x_k)$ such that $p_k = \mu_k \delta_k$. Without any loss of generality, we may assume that the sequence $(\delta_k/\|\delta_k\|)$ converges to some element $v \in S$. Since M is proximally nondegenerate (Definition 2.6, Assertion (*ii*)), one deduces that $v \in \mathrm{cl}(\cup_{\lambda \geq 0} \lambda \partial d_M(x)) = N_M(x)$. But for every k, $p_k = \mu_k \|\delta_k\| (\delta_k/\|\delta_k\|)$, $\|p_k\| = \mu_k \|\delta_k\|$, hence the sequence $(\mu_k \|\delta_k\|)$ converges to $\|p\|$. At the limit, when $k \to \infty$, we get:

$$p = \|p\| v \in N_M(x).$$

Hence $(x, p) \in \mathrm{G}(N_M)$.$\square$

Proof of (*b*). We show hereafter that closed convex subsets of $\mathbb{R}^n$ and closed C^p submanifolds of $\mathbb{R}^n$ are proximally smooth, and then that proximally smooth and epi-Lipschitzian subsets of $\mathbb{R}^n$ are proximally nondegenerate.

Let M be a closed convex subset of $\mathbb{R}^n$ [resp. a closed C^p submanifolds of $\mathbb{R}^n$], then it is proximally smooth. Indeed the distance function d_M is differentiable on $\mathbb{R}^n \setminus M$ [resp. on a neighborhood of M].

Let $M \subset \mathbb{R}^n$ be nonempty and proximally smooth, then it is proximally nondegenerate. Indeed, let $\overline{x} \in M$, then there is $\alpha > 0$ such that d_M is differentiable on $B(\overline{x}, \alpha) \setminus M$. Let $x \in B(\overline{x}, \alpha) \setminus M$, from Clarke [8] (Proposition 2.5.4), there is a unique $p(x) \in M$ such that:

$$\|x - p(x)\| = d_M(x) \text{ and } \nabla d_M(x) = (x - p(x))/\|x - p(x)\|.$$

Noticing that the single-valued map $x \mapsto p(x)$ is continuous, one deduces that ∇d_M is continuous on $B(\overline{x}, \alpha) \setminus M$, hence:

$$\forall x \in B(\overline{x}, \alpha) \setminus M,\ \partial d_M(x) = \{\nabla d_M(x)\} \subset S.$$

Consequently:

$$\forall x \in B(\overline{x}, \alpha) \setminus M,\ \forall p \in \partial d_M(x),\ \|p\| \geq 1.$$

Then the set M clearly satisfies Assertion (*i*) of Definition 2.6. We now show that it satisfies Assertion (*ii*). Consider a sequence (x_k, δ_k) in $(\mathbb{R}^n \setminus M) \times \mathbb{R}^n$, such that $\delta_k \in \partial d_M(x_k)$ for every k, and assume that (x_k) converges to some $x \in M$ and $\delta_k/\|\delta_k\|$ converges to some $p \in S$. Recalling that $\partial d_M(x') \subset \overline{B}(0, 1)$ for every $x' \in \mathbb{R}^n$ (since d_M is 1-Lipschitzian), without any loss of generality, we may assume that the sequence (δ_k) converges to some element $\delta \in \mathbb{R}^n$. But from above $\|\delta_k\| \geq 1$ for k large enough, which implies that $\|\delta\| \geq 1$, and that $p = \lim \delta_k/\|\delta_k\| = \delta/\|\delta\| \in (1/\|\delta\|)\partial d_M(x) \subset N_M(x)$.

Let $M \subset \mathbb{R}^n$ be epi-Lipschitzian, then it is proximally nondegenerate. Indeed, let $\overline{x} \in M$ and let $\Delta_M = d_M - d_{\mathbb{R}^n \setminus M}$ (assume that $M \neq \emptyset$ and $M \neq \mathbb{R}^n$). From [13], $0 \notin \partial \Delta_M(\overline{x})$, hence there is $\alpha > 0$ such that $\overline{B}(0, \alpha) \cap \partial \Delta_M(\overline{x}) = \emptyset$. Since the correspondence $\partial \Delta_M$ is u.s.c., there is $\varepsilon \in (0, \alpha]$ such that $\partial \Delta_M(B(\overline{x}, \varepsilon)) \subset \mathbb{R}^n \setminus \overline{B}(0, \alpha)$. Consequently:

$$\forall x \in B(\overline{x}, \varepsilon) \setminus M, \, \forall p \in \partial d_M(x) = \partial \Delta_M(x), \, \|p\| \geq \varepsilon.$$

This implies, as above, that M is proximally nondegenerate.□

6.2 More on the Euler characteristic

We now relate our definition of the Euler characteristic with the classical one from algebraic topology (see, for example, Dold [16]). Let M be a topological space and let $(H_i M)_{i \in \mathbb{N}}$ be the [singular] homology groups of M. If all the homology groups of M are finitely generated, and if $\mathrm{rank}(H_i M)$ are, but a finite number, equal to zero, one defines the Euler characteristic of M by:

$$\chi_{top}(M) = \sum_{i \in \mathbb{N}} (-1)^i \mathrm{rank}(H_i M).$$

We first consider the class of epi-Lipschitzian sets.

Proposition 6.4 *Let M be a nonempty, compact, epi-Lipschitzian subset of $\mathbb{R}^n$, then:*

$$\chi(M) = \chi_{top}(M).$$

Proof of Proposition 6.4. From Theorem 4.1, let (M_k) be a smooth approximation of M such that M_k and M are homeomorphic for every k. We thus deduce (see, for example, Dold [16]) that:

$$\chi_{top}(M) = \chi_{top}(M_k) \text{ for every } k.$$

Since M_k is a compact differentiable submanifold with a boundary of $\mathbb{R}^n$, of class C^1 and of full dimension, from Milnor ([22], page 36) and Definition 2.3, we get:

$$\chi_{top}(M_k) = \deg(g_{M_k}, \mathrm{int}\, M_k, 0) = \chi(M_k), \text{ where } g_{M_k} \text{ is a Gauss map of } M.$$

Recalling, from Proposition 4.1, that:

$$\chi(M_k) = \chi(M) \text{ for } k \text{ large enough.}$$

From the above equalities, we deduce that $\chi_{top}(M) = \chi(M)$.□

We now consider the class $\overline{\mathcal{L}}$. The following remark shows that the topological Euler characteristic may not be well defined for this class of sets. The following proposition shows that our definition of the Euler characteristic and the topological one coincide in the case of proximally smooth sets. The more general case of proximally nondegenerate sets is considered in [15].

Remark 1. Under the assumptions of Theorem 3.2, the (topological) Euler characteristic $\chi_{top}(M)$ of M may not be defined. Consider the following example from Dold [16] (V.4.12.3):

$$M = \cup_{n=0}^{\infty} S((\frac{1}{2^n}, 0), \frac{1}{2^n}).$$

Then the homology groups of M are not finitely generated, and its (topological) Euler characteristic $\chi_{top}(M)$ is not well defined. However, the set M belongs to $\overline{\mathcal{L}}$ and satisfies the assumptions of Theorem 3.2. Indeed, consider the sequence:

$$M_k = \{x \in \mathbb{R}^n | d_M(x) \leq \frac{1}{2^{k+3}}\}.$$

Notice that the sequence (M_k) converges normally to M, and that, for every k, M_k is epi-Lipschitzian and $\chi(M_k) = -k$ (since M_k is homeomorphic to a disk with $k+1$ holes) and M_k is epi-Lipschitzian (notice that $0 \in \partial d_M(x)$ if and only if $x \in \{(3/2^{i+2}, 0) | i \in \mathbb{N}\}$, which does not meet $\operatorname{bd} M_k$). This example also shows that the class $\overline{\mathcal{L}}$ is strictly larger than the class of proximally nondegenerate sets.

Proposition 6.5 *Let M be a nonempty, compact, proximally smooth subset of $\mathbb{R}^n$, then, for $\varepsilon > 0$ small enough, M is a deformation retract of $\overline{B}(M, \varepsilon)$ and:*

$$\chi(M) = \chi_{top}(M).$$

Proof of Proposition 6.5. Since M is compact and proximally smooth, then the distance function d_M is differentiable on $\overline{B}(M, \varepsilon) \setminus M$ for $\varepsilon > 0$ small enough. Thus, from Clarke [8], for every $x \in B(M, \varepsilon) \setminus M$, there is a unique projection $p_M(x)$ of x onto M, i.e.:

$$p(x) \in M \text{ and } \|x - p_M(x)\| = d_M(x).$$

We define the map $H : [0,1] \times \overline{B}(M, \varepsilon) \to \mathbb{R}^n$ by:

$$H(t, x) = (1-t)x + tp_M(x).$$

One checks that:

$$d_M(H(t,x)) = \|H(t,x) - p_M(x)\| = (1-t)\|x - p_M(x)\| = (1-t)d_M(x).$$

Hence, H is a map from $[0,1] \times \overline{B}(M, \varepsilon)$ to $\overline{B}(M, \varepsilon)$, and it is continuous since p_M is continuous. Thus, H defines a continuous deformation retraction from the epi-Lipschitzian set $\overline{B}(M, \varepsilon)$ to M, which implies (see, for example, Dold [16]) that:

$$\chi_{top}(M) = \chi_{top}(\overline{B}(M, \varepsilon)).$$

From Proposition 6.4, recalling that $\overline{B}(M, \varepsilon)$ is epi-Lipschitzian from Proposition 2.5, we get:

$$\chi(\overline{B}(M, \varepsilon)) = \chi_{top}(\overline{B}(M, \varepsilon)).$$

Recalling that from our definition of the Euler characteristic (Definition 2.7):

$$\chi(M) = \chi(\overline{B}(M, \varepsilon)),$$

we deduce that $\chi(M) = \chi_{top}(M)$.□

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