Fixed-point-like theorems on subspaces

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Abstract

We present several fixed-point theorems whose parameters can be in the set, $G^m(V)$, of all the subspaces of fixed dimension m in a Euclidean space V. The set $G^m(V)$, called the m-Grassmanian manifold of V is a smooth compact manifold but doest not satisfies, in general, properties such as convexity or acyclicity. This prevents from using classical fixed-point theorems of Brouwer (), Kakutani()or Eilenberg-Montgomery.

We shall follow Hirsh-Magill-Mas Colell () and use degree theory in a more basic way and deduce various fixed point theorem which generalize the calssical ones when the variable is a convex domain.

These results were intiated to prove the existence of equilibria in models with incomplete markets. Generalization of the fixed point theorems are also used to study the general equilibrium model with nontransitive preferences.

Key Words: Fixed-point, Grasmanian, degree of maps, transversality, equilibrium theory.

Résumé

Nous présentons ici un théorème de point fixe où certaines variables sont des espaces vectoriels de dimension fixées, inclus dans un espace vectoriel de dimension fini. La principale difficulté est que l'on ne dispose pas sur un tel ensemble -la Grassmanienne- d'hypothèses de convexité ou d'acyclicité permettant d'appliquer des théorèmes classiques de point fixes de type Brouwer. On est ainsi amené à utiliser la théorie du degré, permettant d'obtenir un théorème qui est une alternative aux théorèmes de type Brouwer ou de type Borsuk-Ulam. Enfin, on montre comment ce théorème permet d'obtenir l'existence de pseudo-équilibre (concept permettant de récupérer l'existence générique d'équilibres en marchés incomplets sous certaines hypothèses) ceci dans un cadre très général.

Mots Clés: Points fixes, Grassmanienne, Théorie du degré, transversalité, théorie de l'équilibre.

1 The basic result

1.1 Preliminaries

A correspondence Φ from $X \subset R^n$ to R^m is a map from X to the set of all the subsets of R^m and the graph of Φ, denoted $\mathrm{G}(\Phi)$, is defined by $\mathrm{G}(\Phi) = \{(x, y) \in X \times R^m | y \in \Phi(x)\}$; [1]

The correspondence Φ is said to be upper semicontinuous (u.s.c.), if the set $\{x \in X | \Phi(x) \subset V\}$ is open in X for every open set $V \subset R^m$.

Let V be a finite dimensional Euclidean space and let k be an integer such that $0 \leq k \leq \dim(V)$; then we denote $G^k(V)$ the set consisting of all linear subspaces of V of dimension k, called the (k-)Grassmanian manifold (in V). Then it is known that $G^k(V)$ is a smooth manifold of dimension $k(dim(V) - k)$.

We first state the following result [].

Theorem 1.1 *Let V be finite dimensional Euclidean space and let $M = G^k(V)$, with $0 \leq k \leq dim(V)$. For $i = 1, 2, ..., k$ let $f_i : M \to V$ be a continuous mapping.*

Then, there exist $\bar{E} \in M$ such that

for every $i = 1, 2, ..., k$, $f_i(\bar{E}) \in \bar{E}$.

Remark 1. If we further assume that, for every E, the vectors $f_1(E), \cdots, f_k(E)$ are independent in V, then $\bar{E}$ is a fixed point of the mapping $F : G^k(V) \to G^k(V)$ defined by

$F(E) = \text{span } \{f_1(E), \cdots, f_k(E)\}$.

Indeed, let $\bar{E}$ given by the theorem, then

$F(\bar{E}) = \text{span } \{f_1(\bar{E}), \cdots, f_k(\bar{E})\} \subset \bar{E}$,

and the equality holds from the independence assumption. □

[0] We let $R_+ = \{x \in R | x \geq 0\}$. If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ belong to R^n, we denote $(x|y) = \sum_{i=1}^n x_i y_i$, the scalar product of R^n, $\|x\| = \sqrt{(x|x)}$, the Euclidian norm; we denote $B(x, r) = \{y \in R^n | \ \|x - y\| < r\}$, $\overline{B}(x, r) = \{y \in R^n | \ \|x - y\| \leq r\}$ and $S(x, r) = \{y \in R^n | \ \|x - y\| = r\}$. If $X \subset R^n$, $Y \subset R^n$, and $x \in R^n$, we let $d(x, X) = \inf_{y \in X} \|x - y\|$, we denote $X \setminus Y = \{x \in X | x \notin Y\}$ the set-difference of the sets X and Y, $X + Y = \{x + y | x \in X, y \in Y\}$, the sum of the sets X and Y, $B(X, r) = X + B(0, r)$, $\overline{B}(X, r) = X + \overline{B}(0, r)$, cl$X$ or $\overline{X}$, the closure of X, int$X$, the interior of X, bd$X$ =cl$X \setminus$ int$X$, the boundary of X, $X^\circ = \{y \in R^n | \forall x \in X, \ (y|x) \leq 0\}$, the negative polar cone of X, $X^\perp = \{y \in R^n | \forall x \in X, (x|y) = 0\}$, the orthogonal vector space to X, co X, the convex hull of X.

[1] If Φ and Ψ are two correspondences from $X \subset R^n$ to R^m, the correspondences $\Phi \cap \Psi$, co Φ, are defined, respectively, by $(\Phi \cap \Psi)(x) = \Phi(x) \cap \Psi(x)$, $(\mathrm{co}\,\Phi)(x) = \mathrm{co}\,\Phi(x)$. A map $\varphi : X \to R^m$ is called a selection of Φ, if $\varphi(x) \in \Phi(x)$ for all $x \in X$. If A is a subset of X, we denote $\Phi(A) = \cup_{x \in A} \Phi(x)$ and we define the restriction of Φ to A, denoted $\Phi|_A$, to be the correspondence from A to R^m defined by $\Phi|_A(x) = \Phi(x)$ if $x \in A$.

Remark 2. Note that a continuous mapping $F : G^k(V) \to G^k(V)$ may not have a fixed-point. Indeed, consider $M = G^1(R^2)$ the set of all lines passing through the origin, then the rotation of $\pi/2$ from $G^1(R^2)$ to itself, does not have a fixed-point. So the above result is only a fixed-point like theorem.

Remark 3. The above theorem is still true if we consider only r mappings $f_i : M = G^k(V) \to V$ $(i = 1\cdots, r$ with $r < k$. (Consider $k - r$ arbitrary additional mappings and apply the above theorem to the k new mappings).

However, the theorem is no longer valid, in general, if we consider more than k mappings. Indeed, consider again $M = G^1(R^2)$ the set of all lines passing through the origin, which we suppose are parametrized by the angle θ (modulo π) with the x axis. We define the functions $\hat{f}_i : R \to R^2$ $(i = 1, 2)$ by

$$\hat{f}_1(\theta) = \cos(\theta)(\cos(\theta), \sin(\theta)), \hat{f}_2(\theta) = \sin(\theta)(\cos(\theta), \sin(\theta)).$$

Noticing that $\hat{f}_i(\theta + \pi) = \hat{f}_i(\theta)$ $(i = 1, 2)$, the $\hat{f}_i$ induce mappings $f_i : G^1(R^2) \to R^2$ $(i = 1, 2)$ and, for every $E \in G^1(R^2)$, $E \neq \text{span}\ \{f_1(E), f_2(E)\}$.

□

1.2 Proof of the basic theorem

We define $\psi : G^m(V) \to V^m$ by

$$\psi(E) = (\text{proj}_{E^\perp}\, \psi_1(x), ..., \text{proj}_{E^\perp}\, \psi_m(x))$$

and we define $F : G^m(V) \to G^n(V^m)$ by

$$F(E) = E^\perp \times \cdots \times E^\perp.$$

We then show the theorem by contraposition. If it is not true, then the set $f^{-1}(0)$ is empty, hence 0 is a regular value of f and card $f^{-1}(0) = 0$. Consequently, by Theorem, for every $g \in \mathcal{S}_0(F)$, card $g^{-1}(0) = 0$, (modulo 2). We end the proof by contradicting this last assertion, more precisely, by constructing a mapping $g \in \mathcal{S}_0(F)$, such that card $g^{-1}(0) = 1$. For this, consider a fixed element E in $G^n(V)$, and consider an an orthonormal basis $\{\bar{e}_1, ..., \bar{e}_m\}$ of E. We define the mapping $g : G^m(V) \to V^m$ by

$$g(L) = (\text{proj}_{L^\perp}(\bar{e}_1), ..., \text{proj}_{L^\perp}(\bar{e}_m)).$$

We first notice that $g^{-1}(0) = \{E\}$. Indeed, clearly, $E \in g^{-1}(0)$. Conversely, let $L \in g^{-1}(0)$, then $\text{proj}_{L^\perp}(\bar{e}_1) = \cdots = \text{proj}_{L^\perp}(\bar{e}_m) = 0$. Hence, the (independent) vectors $e_1, \cdots, e_m$ belong to L, a subspace of dimension m. Consequently, L is equal to E, which is the subspace spanned by $e_1, \cdots, e_m$.

We end the proof by showing that 0 is a regular value of the mapping g, or equivalently that the unique element in $g^{-1}(0)$, that is E, is a regular point of the mapping $\alpha : L \to \operatorname{proj}_{F(E)} g(L)$ from the -manifold $X = G^m(V)$ to the -dimensional subspace $F(E) = (E^{\perp})^m$. This again is equivalent to saying that the derivative $D\alpha(E)$ at E is bijective. For this consider a chart (φ, U) of the -manifold $X = G^m(V)$ containing E. From the Appendix we can consider the mapping φ^{-1} from $\varphi^{-1}(U) \subset (E^{\perp})^m$ to U defined by :

$$\varphi^{-1}(U) = E(u) := \operatorname{span}\{e_1+u_1, ..., e_n+u_n\}, \text{for} u = (u_1, \cdots, u_n) \in \varphi^{-1}(U) \subset (E^{\perp})^m.$$

Hence the mapping $go\varphi^{-1}$ is defined on a neighborhood of 0 and

$go\varphi^{-1}(u) = (\operatorname{proj}_{E(u)} e_1, \cdots, \operatorname{proj}_{E(u)} e_m)$.

Then from the lemma below (taking $x = e_1, \cdots, x = e_m$ successively) one deduces that its derivative at 0, $D(go\varphi^{-1})(0)$, is the injection mapping $i : (E^{\perp})^m \to V$ (defined by $i(u) = u$). Hence the derivative $D(\alpha o\varphi^{-1}))(0) = \operatorname{proj}_{F(E)} oD(go\varphi^{-1}))(0)$ is the identity mapping on $F(E) = (E^{\perp})^m$. Hence E is a regular point of g. □

In the end of the proff we have used the following lemma we now state and prove.

Lemma 1.1 *Let $E \in G^n(V)$, let $\{e_1, ..., e_n\}$ be an orthonormal basis of E, for every $u = (u_1, ..., u_n) \in (E^{\perp})^n$ we let $E(u) = span\{e_1 + u_1, ..., e_n + u_n\}$.*

(a) Let $x = \sum_{i=1}^n x_i e_i \in E$ be a given vector and let $h : (E^{\perp})^n \to V$ be defined by $h(u) = proj_{E(u)} x$. Then the mapping h is smooth in a neighborhood of 0 and the derivative of h at 0 is the linear mapping $Dh(0) : (E^{\perp})^n \to V$ defined by

$$Dh(0)(u) = \sum_{i=1}^n x_i u_i, \forall u = (u_i) \in (E^{\perp})^n.$$

(b) Let $\hat{g} : (E^{\perp})^n \to V^m$ be defined by $\hat{g}(u) = (\operatorname{proj}_{E(u)} e_1, \cdots, \operatorname{proj}_{E(u)} e_m)$. Then the mapping $\hat{g}$ is smooth in a neighborhood of 0 and the derivative of h at 0 is the linear injection mapping $i : (E^{\perp})^m \to V$ (defined by $i(u) = u$).

Proof of Lemma 1.4. *Part (a)* There exists $\lambda(u) = (\lambda_j(u))_{j=1,\dots,n} \in R^n$ such that

$$proj_{E(u)} x = \sum_{j=1}^n \lambda_j (e_j + u_j).$$

We have $x - proj_{E(u)} x \in E(u)^{\perp}$ or equivalently for every $i = 1, ..., n$

$$(-x + \sum_{j}^{n} \lambda_j(u)(e_j + u_j)) \cdot (e_i + u_i) = 0$$

which can be written for every $i = 1, ..., n$

$$\sum_{j}^{n} \lambda_j(u)(e_j + u_j) \cdot (e_i + u_i) = x \cdot e_i$$

or

$$(Id_{R^n} + G(u))\lambda(u) = (x_1, ..., x_n)$$

where $G(u) = (u_i \cdot u_j)_{i,j=1,...,n}$. Since $G(0) = 0$, G being continuous and $GLn(n)$ being open in $M(n, n)$, for U smaller enougn, $(Id_{R^n} + G(u))$ is invertible. We want to prove that

$$Dg(0)(u) = \sum_{j=1}^{n} u_i \cdot x_j$$

or equivalently

$$lim_{t \to O^+}(g(tu) - g(0))/t = \sum_{j=1}^{n} u_i \cdot x_j.$$

But

$$(g(tu) - g(0))/t = \sum_{j=1}^{n}(\lambda_j(tu) - \lambda_j(0))/te_j + \sum_{j=1}^{n} \lambda_j(tu)u_j$$

and

$$\begin{aligned}(\lambda_j(tu) - \lambda_j(0))/t &= (1/t)((Id + G(tu))^{-1} - Id)(x_1, ..., x_n)\\ &= (1/t)(Id + G(tu)^{-1}(Id - (Id + G(tu))(x_1, ..., x_n)\\ &= -(1/t)(Id + G(tu)^{-1}(G(tu))(x_1, ..., x_n)\\ &= -(1/t)(Id + t^2G(tu)^{-1}(t^2G(tu))(x_1, ..., x_n)\end{aligned}$$

since $G(tu) = t^2G(u)$.

So $\lim_{t\to 0}(\lambda_j(tu) - \lambda_j(u))/t = 0$ and $\lim_{t\to 0}(g(tu) - g(u))/t = \lim_{t\to 0} \sum_{j=1}^{n} \lambda_j(0)u_j$. Since $\lambda_j(0) = x_j$, we have what we wanted.

□

2 More general results

The purpose of this paper is to prove the following theorem:[2]5Il serait bon de regrouper les deux theoremes suivant en un seul en considerant par exemples les

[2]

Theorem 2.1 *(ancien) Let V a finite dimensional Euclidean space, for $i = 1, ..., n$, let C_i be a nonempty, compact, convex subset of some Euclidean space, and let $X = \prod_{i=1}^{n} C_i \times G^m(V)$ with $0 \le m \le dim(V)$.*

For $i = 1, ..., n$, let φ_i be a correspondence from X to C_i, which is lower semicontinuous and convex valued (possibly empty-valued), and for $j = 1, ..., m$ let $\psi_j : X \to V$ be a continuous mapping.

Then, there exist $\bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n, \bar{E}) \in X$ such that

(a) for every $i = 1, ..., n$, [either $\bar{x}_i \in \varphi_i(\bar{x})$ or $\varphi_i(\bar{x}) = \emptyset$];

(b) for every $j = 1, ..., m$ $\psi_j(\bar{x}) \in \bar{E}$.

correspondances approximables).

Theorem 2.2 *Let V a finite dimensional Euclidean space, for $i = 1, ..., n$, let C_i be a nonempty, compact, convex subset of some Euclidean space, and let $X = \prod_{i=1}^{n} C_i \times G^m(V)$ with $0 \leq m \leq dim(V)$.*

For $i = 1, ..., n$, let φ_i be a correspondence from X to C_i, and for $j = 1, ..., m$ let ψ_j be a correspondence from X to V. We assume that the φ_i and the ψ_j are lower semicontinuous and convex valued (possibly empty-valued).

Then, there exist $\bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n, \bar{E}) \in X$ such that

(a) for every $i = 1, ..., n$, $[$either $\bar{x}_i \in \varphi_i(\bar{x})$ or $\varphi_i(\bar{x}) = \emptyset]$;

(b) for every $j = 1, ..., m$ $[$either $\bar{E} \cap \psi_j(\bar{x}) \neq \emptyset$ or $\psi_j(\bar{x}) = \emptyset]$.

Theorem 2.3 *Let V a finite dimensional Euclidean space, for $i = 1, ..., n$, let C_i be a nonempty, compact, convex subset of some Euclidean space, and let $X = \prod_{i=1}^{n} C_i \times G^m(V)$ with $0 \leq m \leq dim(V)$.*

For $i = 1, ..., n$, let φ_i be a correspondence from X to C_i, and for $j = 1, ..., m$ let ψ_j be a correspondence from X to V. We assume that the φ_i and the ψ_j are upper semicontinuous and closed, convex valued (possibly empty-valued).

Then, there exist $\bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n, \bar{E}) \in X$ such that

(a) for every $i = 1, ..., n$, $[$either $\bar{x}_i \in \varphi_i(\bar{x})$ or $\varphi_i(\bar{x}) = \emptyset]$;

(b) for every $j = 1, ..., m$ $[$either $\bar{E} \cap \psi_j(\bar{x}) \neq \emptyset$ or $\psi_j(\bar{x}) = \emptyset]$.

2.1 Some consequences

Corollary 2.1 *Let V a finite dimensional Euclidean space, , and let $X = G^m(V)$ with $0 \leq m \leq dim(V)$.*

For for $j = 1, ..., m$ let ψ_j be a correspondence from X to V. We assume that the ψ_j are upper semicontinuous with **nonempty** *closed, convex valued .*

Then, there exist $\bar{E} \in X$ such that

for every $j = 1, ..., m$ $\bar{E} \cap \psi_j(\bar{E}) \neq \emptyset$.

We now generalize the previous result by considering the pruct...

Theorem 2.4 *Let V_1 and V_2 be finite dimensional Euclidean spaces and let $X = G^{m_1}(V_1) \times G^{m_2}(V_2)$, with $0 \leq m_1 \leq dim(V_1)$ and $0 \leq m_2 \leq dim(V_2)$. For $i = 1, 2, ..., m_1$ let $\psi_i : X \to V_1$ be a continuous mapping and for $j = 1, 2, ..., m_2$ let $\varphi_j : X \to V_2$ be a continuous mapping*

Then, there exist $\bar{x} = (\bar{x}_1, \bar{x}_2) \in X$ such that

(a) for every $i = 1, 2, ..., m_1$ we have $\psi_i(\bar{x}) \in \bar{x}_1$;

(b) for every $j = 1, 2, ..., m_2$ we have $\varphi_j(\bar{x}) \in \bar{x}_2$.

Proof of Theorem We define $f : G^{m_1}(V_1) \times G^{m_1}(V_1) \to (V_1 \times \{0\})^{m_1} \times (\{0\} \times V_2)^{m_2}$ by $f(x, y) = (p_{x^\perp}\psi_1(x) \times \{0\}, ..., p_{x^\perp}\psi_{m_1}(x) \times \{0\}, \{0\} \times p_{y^\perp}\varphi_1(x), ..., \{0\} \times p_{y^\perp}\varphi_{m_2}(x))$

and we define $F : G^{m_1}(V_1) \times G^{m_1}(V_1) \to G^{m_1+m_2}((V_1 \times \{0\})^{m_1} \times (\{0\} \times V_2)^{m_2})$ by $F(x, y) = (x^\perp \times \{0\}, ..., x^\perp \times \{0\}, \{0\} \times y^\perp, ..., \{0\} \times y^\perp)$.

We first prove that $Deg_{G^{m_1}(V_1)\times G^{m_1}(V_2)}(F) = 1$ which implies, since $f \in \mathcal{C}_{X,F}$, that there exists $(\bar{x}, \bar{y}) \in G^{m_1}(V_1) \times G^{m_1}(V_2)$ such that $f(\bar{x}, \bar{y}) = 0$.

Theorem 2.5 *Let V be finite dimensional Euclidean space, let C be a nonempty, compact, convex subset of some Euclidean space, and let $X = C \times G^m(V)$. Let φ be a mapping from X to C, which is continuous, and for $j = 1, ..., m$ let $\psi_j : X \to V$ be a continuous mapping.*

Then, there exist $\bar{x} = (\bar{x}_1, \bar{x}_2) \in X$ such that

(a) $\bar{x}_1 \in \varphi(\bar{x})$

(b) for every $j = 1, ..., m$ $\psi_j(\bar{x}) \in \bar{x}_2$.

We can give a direct proof of Corollary 3.3 with Corollary 3.2.

Proof of Corollary 2.3 It is clearly sufficient to prove the theorem in the case where C is a set homeomorphic to a convex compact subset of R^p. Then we consider the compactification C_∞ of the set $A = \{(x_1, ..., x_p) \in R^p_+ \mid x_p = -1\}$. We now define $\varphi_1' : G^1(R^p) \times G^m(V) \to C_\infty$ and for $j = 1, ..., m$ the mappings $\psi_j' : G^1(R^p) \times G^m(V) \to V$ by:

$[\psi_j'(E, E') = \psi_j(E \cap A, E'))$ and $\varphi_1'(E, E') = \varphi_1(E \cap A, E')]$ if $A \cap E \neq \emptyset$ and $[\psi_j'(E, E') = \psi_j(\infty, E'))$ and $\varphi_1'(E, E') = \varphi_1(\infty, E')]$ otherwise.

These functions are continuous. Hence by the latest theorem, there is $(\bar{E}, \bar{E}')$ with $\psi_j'(\bar{E}, \bar{E}') \in \bar{E}'$ and $\varphi_1(\bar{E}, \bar{E}') \in \bar{E}$.

First, it implies that $\bar{E} \cap A \neq \emptyset$. Otherwise, we would have $\varphi_1(\infty, \bar{E}') \in \bar{E}$ and $\varphi_1(\infty, \bar{E}') \in A$ a contradiction.

Then we have $\psi_j(\bar{E} \cap A, \bar{E}') \in \bar{E}'$ $(j = 1, ..., J)$ and $\varphi_1(\bar{E} \cap A, \bar{E}') \in \bar{E}$. Since $\varphi_1(\bar{E} \cap A, \bar{E}') \in C$ we obtain $\varphi_1(\bar{E} \cap A, \bar{E}') \in \bar{E} \cap A$. Finally,if we let $\{\bar{x}_1\} = (\bar{E} \cap A)$ then the element $(\bar{x}_1, \bar{E}')$ satisfies:

(a) $\bar{x}_1 \in \varphi_1(\bar{x}_1, \bar{E}')$

(b) for every $j = 1, ..., m$ $\psi_j(\bar{x}_1, \bar{E}') \in E'$.

Encore un theoreme

Theorem 2.6 *Let V be finite dimensional Euclidean space, let C be nonempty, compact, convex subset of some Euclidean space, and let $X = C \times G^m(V)$.*

Let φ be a correspondence from X to C, which is upper semicontinuous and nonempty, convex,compact valued , and for $j = 1, ..., m$ let $\psi_j : X \to V$ be a continuous mapping.

Then, there exist $\bar{x} = (\bar{x}_1, \bar{x}_2) \in X$ such that

(a) $\bar{x}_1 \in \varphi(\bar{x})$

(b) for every $j = 1, ..., m$ $\psi_j(\bar{x}) \in \bar{x}_2$.

Proof of Theorem From Cellina (), for every $\epsilon > 0$ there exists a continuous mapping $\varphi_1^\epsilon : X \to C$ such that for every $x \in X$ we have $Gr(\varphi_1^\epsilon) \subset Gr(\varphi_1) + \epsilon B(0,1)$. We apply Corollary 3.2 to the mappings φ_1^ϵ and ψ_j $j = 1, ..., m$. Then we obtain $(\bar{x}_1^\epsilon, \bar{E}^{\epsilon'}) \in X$ that satisfies:

(a) $\bar{x}_1^\epsilon \in \varphi_1^\epsilon(\bar{x}_1^\epsilon, \bar{E}^{\epsilon'})$

(b) for every $j = 1, ..., m$ $\psi_j(\bar{x}_1^\epsilon, \bar{E}^{\epsilon'}) \in E^{\epsilon'}$.

Since X is compact we can suppose without any loss of generality that the sequence $(\bar{x}_1^\epsilon, \bar{E}^{\epsilon'})$ converges, when ϵ converges to 0, to an element $(\bar{x}_1, \bar{E})$ that satisfies the conclusion of theorem 2.2.

Proof of Theorem 1.1 For every $i = 1, ..., n$, let $U_i = \{x \in C_1 \times ... \times C_n \mid$ there exists $E \in G^J(V), \varphi_i(x, E) \neq \emptyset\}$, then $\varphi \mid U_i \times G^J(V) : U_i \times G^J(V) \to C_i$ is a convex valued correspondence having an open graph. Let $f_i : U_i \times G^j(V) \to C_i$ be a continuous function such that $f_i(x, E) \in \varphi_i(x, E)$ fo every $(x, E) \in U_i \times G^J(V)$ [see Michael]. For $i = 1, ..., n$, define correspondences $\varphi_i : X \to C_i$ by $\varphi_i(x, E) = f_i(x, E)$ if $x \in U_i$ and $\varphi_i(x, E) = C_i$ otherwise. For every i, φ_i is non-empty, convex valued and upper-semicontinous. Then we can apply Theorem 4 to $\varphi : X \to \prod C_i$ define by $\varphi(x) = \prod_{i=1}^n \varphi_i(x)$ and to the mappings ψ_j.

Then there exists $(\bar{x}, \bar{E}) = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n, \bar{E}) \in X$ such that

(a) $\bar{x} \in \varphi(\bar{x})$;

(b) for every $j = 1, ..., J$, $\psi_j(\bar{x}) \in \bar{E}$.

By construction, $(\bar{x}, \bar{E})$ satisfies the conclusion of the Theorem 1.1.

□

3 Appendix

3.1 The grasmanian manifold $G^k(V)$

Let V be a Euclidean space and let k be an integer such that $0 \leq k \leq \dim(V)$; then we denote by $G^k(V)$ the set consisting of all linear subspaces of V of dimension k, which is called the k-Grassmanian manifold of V.

Clearly, $G^k(V)$ is a metric space for the distance d: $G^k(V) \times G^k(V) \to R^+$ defined (via the Haussdorff distance) as follows by

$$\mathrm{d}(L_1, L_2) = \delta(L_1 \cap B, L_2 \cap B),$$

where δ denotes the Hausdorff distance, and B the closed unit ball of V.[3]

For each element $L \in G^m(V)$, we can define a basis of open neighborhood as follows: for each $L \in G^m(V)$, let $(e_1, ..., e_m$ an orthonormal basis of L. Then we define $V_{\epsilon, e_1, ..., e_m}(L) := \{L^{'} = span(e_1^{'}, ..., e_k^{'}) \mid (e_1^{'}, ..., e_k^{'})$ is orthonormal and for every $i = 1, ..., k$ $\|e_i^{'} - e_i\| < \epsilon\}$. This topology spanned by these open neighborhoods has several important properties:

1) $G^m(V)$ is compact for this topology.

2) The mapping p define from $V \times G^m(V)$ to V by $p(x, E) = proj_E(x)$ is continuous.

3) The set $\{(x, E) \in V \times G^m(V) \mid x \in E\}$ is closed.

4) The mapping p_1 define from $V \times G^m(V)$ to V by $p_1(x, E) = proj_{E^{\perp}}(x)$ is continuous.

Proof 1) We consider a sequence E^p of element of $G^m(V)$ and $(e_1^p, ..., e_m^p)$ an orthonormal basis of E^p. Then, we can suppose, without any loss of generality, that for every $i = 1, ..., m$ the sequence e_i^p converges towards an element $\bar{e}_i$. Then $(\bar{e}_1, ..., \bar{e}_m)$ is orthonormal, and if we define $\bar{E} = span(\bar{e}_1, ..., \bar{e}_m)$ it is immediat that E^p converges towards $\bar{E}$ for the topology defined just before.

2) We now consider a sequence (x^p, E^p) of element of $V \times G^m(V)$ converging towards $(\bar{x}, \bar{E})$. Then we can write $x^p = \sum_{i=1}^m (x^p \cdot e_i^p) e_i^p + x^{'p}$, where $(e_1^p, ..., e_m^p)$ is an orthonormal system that spans E^p and where $x^{'p} \in E^{\perp}$. We can suppose without any loss of generality that for every $i = 1, ..., m$ the sequence e_i^p converges towards an element $\bar{e}_i$ and then the sequence $x^{'p}$ converges to $x^{'}$. Then we obtain $\bar{x} = \sum_{i=1}^m (\bar{x} \cdot \bar{e}_i + \bar{x}^{'}$ which proves that $p(x^p, E^p)$ converges to $p(\bar{x}, \bar{E})$.

3) We notice that $\{(x, E) \in V \times G^m(V) \mid x \in E\} = \{x \mid x = proj_E(x)\}$ which entails the result.

4) It is a consequence of 2). Indeed we can notice that $proj_E(x) + proj_{E^{\perp}}(x) = x$.

Remark: we ca also define this topology by the finest topology that makes the mapping p continuous. But it is then less obvious to obtain compacity. Another way to obtain compacity is to define the topology by defining a quotient topology:

Let $O^m(V)$ the set of m orthonormal vectors of R^n. We say that two elements $(x_1, ..., x_k)$ and $(y_1, ..., y_k)$ of $O^m(V)$ are equivalent (i.e $(x_1, ..., x_m) \sim (y_1, ..., y_m)$) if span$(x_1, ..., x_m)$= span$(y_1, ..., y_m)$. Since $G^m(V) = O^m/\sim$, we can define the quotient topology on $G^m(V)$ as the finest topology that makes the canonic sur-

[3]If K_1 and K_2 are two compact subsets of V, we recall that

$$\delta(K_1, K_2) = \max(\sup d(x, K_1)_{x \in K_2}, \sup d(x^{'}, K_2)_{x^{'} \in K_1}.$$

jection from $O^m(V)$ to $O^m/\sim$ continuous. We can also define this topology as the finest topology that make continuous the mapping f from $O^m(V)$ to $G^m(V)$ define by $f(u_1, ..., u_m) = span(u_1, ..., u_m)$. We can easily show that this topology and the previous one are the same. Now, the compactness of $G^m(V)$ is an immediat consequence of the compactness of $O^m(V)$.

3.2 The degree of a map from an m-manifold M to $G^m(V)$

In the following, we let M be a smooth[4] manifold of dimension m, let V be a Euclidean space such that $\dim(V) \geq m$, and we let $\mathcal{C}^\infty(X, G^n(V))$ [5]be the set of continuous mapping from M to $G^m(V)$, the m-Grassmanian manifold, whose $\mathcal{C}^\infty$-manifold structure has been defined in the previous section 4.1. (peut etre a enlever)

In the whole section we let F in $\mathcal{C}^\infty(M, G^m(V))$, we let $\mathcal{S}(F)$ be the set of smooth selection f of F, that is $f : M \to V$ is smooth, and for all x in X, $f(x) \in F(x)$, and we recall that $G(F) = \{(x, y) \in M \times V \mid y \in F(x)\}$ denotes the graph of F.

We say that 0 is a regular value of the mapping $f \in \mathcal{S}(F)$ if for every $\bar{x} \in f^{-1}(0)$ the linear mapping $\text{proj}_{F(\bar{x})} \circ Df(\bar{x})$, from T_xM to $F(\bar{x})$, is bijective. We notice that $\bar{x} \in f^{-1}(0)$ satisfying the previous condition if (and only if) it is a regular point (in the classical sense, cf., for example, Hirsh (1976)) of the mapping $\text{proj}_{F(\bar{x})} \circ f$, from the m-manifold M to the m-dimensional Euclidean space $F(\bar{x})$. We shall give later an equivalent condition of regularity in terms of transversality condition. In the following, we denote $\mathcal{S}_0(F)$ the set of mappings $f \in \mathcal{S}(F)$ such that 0 is a regular value of f in the above sense.[6]

In this section we prove the following Theorem, which is the key tool for the proof of the fixed-point theorems of this paper.[7]

Theorem 3.1 *Let M a compact smooth manifold of dimension m, then for every f_1, f_2 in $\mathcal{S}_0(F)$, the sets $f_1^{-1}(0)$ and $f_2^{-1}(0)$ are finite and: card $f_1^{-1}(0) \equiv$ card $f_2^{-1}(0)$ (modulo 2).*

The proof of Theorem 4.1 is a consequence of the following lemma.

Lemma 3.1 *(a) If $f \in \mathcal{S}_0(F)$. then $f^{-1}(0)$ is finite.*

[4]that is of class C^∞. Verifier si C^1 suffit.

[5]Peut on passer a C^0?

[6]Note that f is a mapping from an m-manifold to a finite dimensional vector space V whose dimension is greater or equal to m. Considering the projection mapping $\text{proj}_{F(\bar{x})}$ allows us to get mappings between manifolds of the same dimension.

[7]At this stage it is worth noticing that we are half way from the definition of the degree of F (modulo 2) and in the following we shall only use the above theorem. It only suffices to show that the set $\mathcal{S}_0(F)$ is nonempty to be able to define deg$F$ = card $f^{-1}(0)$ (modulo 2).

(b) If $g \in \mathcal{S}(F)$ is close enough to f for the C^1 norm, [8] *then $g \in \mathcal{S}_0(F)$ and card $f^{-1}(0) \equiv$ card $g^{-1}(0)$ (modulo 2).*

Proof of Lemma 4.1. Step 1 Let $\bar{x} \in f^{-1}(0)$, we define the mapping H on a neighborhood of $(f, \bar{x})$ by $H(g, x) :=\text{proj}_{F(\bar{x})} g(x)$. Clearly, we have $H(f, \bar{x}) = 0$, and the derivative (with respect to the second variable x) $D_x H(f, \bar{x}) =\text{proj}_{F(\bar{x})} Df(\bar{x})$ is bijective since $\bar{x}$ is a regular point of f. Consequently, from the implicit function theorem, there exist $\varepsilon, \varepsilon'$ and a continuous map $x(.) : B(f, \varepsilon) \to M$, such that, for every $g \in B(f, \varepsilon)$, $x = x(g)$ is the unique point in $B(\bar{x}, \varepsilon') \cap M$, satisfying $H(g, x(g)) = 0$, that is $\text{proj}_{F(\bar{x})} g(x(g)) = 0$, or equivalently $g(x(g)) \in F(\bar{x})^{\perp}$. Recalling that $g(x(g)) \in F(x(g))$ we deduce that

$$g(x(g)) \in F(\bar{x})^{\perp} \cap F(x(g)),$$

and we now show that, taking eventually ε' smaller, this set is reduced to $\{0\}$, which implies that $g^{-1}(0) \cap B(\bar{x}, \varepsilon) \cap M = \{x(g)\}$.

For this we shall show that, for every $\varepsilon' > 0$ and $x \in B(\bar{x}, \varepsilon)$, one has $F(\bar{x})^{\perp} \cap F(x) = \{0\}$. If not there is a sequence $(x^{\nu}) \to \bar{x}$ and a sequence (v^{ν}) of norm 1, such that $v^{\nu} \in F(\bar{x})^{\perp} \cap F(x^{\nu})$. Without any loss of generality we can suppose that the sequence (v^{ν}) converges to some element $\bar{v}$ of norm 1 and one deduces that $0 \neq \bar{v} \in F(\bar{x})^{\perp} \cap F(\bar{x})$, a contradiction.

Step 2 From above, the set $f^{-1}(0)$ has only isolated points in the compact M and so is finite set$\{x_1, \cdots, x_k\}$, which proves part a). From Step 1, we proved that there exists neighborhoods U_i of x_i $(i = 1 \cdots, k)$ and a neighborhood U of f such that $g^{-1}(0) \cap U_k$ consists of a single point $x_k(g)$. Hence there is some $m > 0$ such that $\|f(x)\| \geq 0$ for x in the compact set $M \setminus \cup_i U_i$. Taking g in $U \cap B(f, m/2)$, one deduces that $g^{-1}(0) = \{x_1(g), \cdots, x_k(g)\}$ which proves part b. □

Lemma 3.2 *Let $f : M \to V$ be a smooth selection of F, we let $\hat{f} : M \to G(F)$ be defined by $\hat{f}(x) = (x, f(x))$. Then the following assertions (i), (ii) and (iii) are equivalent :*

(i) 0 is a regular value of f, that is, for every $x \in f^{-1}(0)$ the map $\text{proj}_F(x) \circ Df(x)$ is bijective;

(ii) for every $x \in f^{-1}(0)$, $Im Df(x) = F(x)$;

(iii) $\hat{f}$ is transverse to $M \times \{0\}$,[9] *in the sense that:*

$$D\hat{f}(x)(T_x M) + T_{\hat{f}(x)}(M \times \{0\}) = T_{\hat{f}(x)} G(F) \forall x \in M, \hat{f}(x) \in M \times \{0\}.$$

[8] to be defined here

[9] $\hat{f}$ is transverse to $\hat{I}d$ defined by $\hat{i}(x) = (x, 0)$ in the sense that

$$\forall x \in M \mid \hat{f}(x) = \hat{I}d(x) \ , T_{(x,0)} G(F) = D\hat{f}(x) T_x M \oplus D\hat{I}d_0(x) T_x M.$$

Proof of Lemma 4.2 We first notice that the above transversality condition is equivalent to

$$D\hat{f}(x)(T_xM) + T_xM \times \{0\} = T_{(x,0)}G(F) \forall x \in M, f(x) = 0.$$

and that

$D\hat{f}(x)T_xM = \{(v, Df(x)v) \mid v \in T_x(M)\} = T_x(M)\times$ Im $Df(x)$.

Hence

$D\hat{f}(x)T_xM + D\hat{I}d(x)T_xM = .$ (1)

We now show that

$$T_{(x,0)}G(F) = T_xM \times F(x).(2)$$

Indeed, if (v, w) is in $T_{(x,0)}G(F))$ then there exists a continuous mapping $(x(t), y(t))$ from I to $G(F)$, where I is an open interval of R containg 0, such that $(x(0), y(0)) = (0, 0)$ and

$$\lim_{t\to 0}(\phi(x(t)) - \phi(x))/t = v \ (3)$$

$$\lim_{t\to 0} y(t)/t = w \ (4)$$

where (ϕ, U) is a local chart of M containing x. From (1) we get that $v \in T_xM$. Besides, if we let $\epsilon(t) = w - y(t)/t$ then one has $t(w - \epsilon(t)) \in F(x(t))$; since $F(x(t))$ is a vector space, $w - \epsilon(t) \in F(x(t))$, and making $t \to 0$, $w \in F(x)$.

Conversely, if $(v, w) \in T_xM \times F(x)$, then there exists a continuous mapping $x(t)$ from I to M, where I is an open intervalle of R containg 0, such that $x(0) = x$ and

$$\lim_{t\to O}(\phi(x(t)) - \phi(x))/t = v.$$

where (ϕ, U) is a local chart of M contaning x. Then we consider $y(t) = t\text{proj}_{F(x(t))}(w)$, and we obtain a continuous mapping $(x(t), y(t))$ from I to $G(F)$, where I is an open intervalle of R containg 0, such that $(x(0), y(0)) = (0, 0)$ and

$$\lim_{t\to O}(\phi(x(t)) - \phi(x))/t = v.$$

$$\lim_{t\to O} y(t)/t = w.$$

which prove that (v, w) is in $T_{(x,0)}G(F))$. This ends the proof of (2). In view of (1) and (2) the condition (iii) holds if and only if for every $x \in M$ such that $\hat{f}(x) = \hat{I}d(x)$ one has $F(x) = ImDf(x)$, that is if and only if the condition (ii) holds.

Now, we prove that for every x such that $f(x) = 0$ then Im$Df(x) \subset F(x)$. Indeed, there exists a continuous mapping $x(t)$ from I to M, where I is an open interval of R containg 0, such that $x(0) = 0$ and

$$\lim_{t\to 0}(f(x(t)) - f(x))/t = u$$

where $f(x) = 0$. So, if we let $u(t) = f(x(t)/t$, one has $tu(t) \in F(x(t))$; since $F(x(t))$ is a vector space, $u(t) \in F(x(t))$, and making $t \to 0$, $u \in F(x)$.

To finish, we can notice that if $x \in M$ is such that $f(x) = 0$ then saying that O is a regular value of f is saying that $\mathrm{Im}Df(x) = F(x)$ because dim $T_x(M)$=dim$F(x)$, which ends the proof.

□

We then prove the following lemma:

Lemma 3.3 *For every f_1, f_2 in $\mathcal{S}_0(F)$, for every $\varepsilon > 0$, there exists a smooth mapping $H : [0,1] \times M \to V$ such that the mapping $\hat{H} : [0,1] \times M \to G(F)$ defined by $\hat{H}(t,x) = (x, H(t,x))$, satisfies*

(i) $\hat{H}$ is transverse to $M \times \{0\}$;

(ii) the restriction of $\hat{H}$ to $\partial([0,1] \times M)$, denoted $H_{|\partial([0,1]\times M)}$, is transverse to $M \times \{0\}$.

(iii) $\|H(0,.) - f_1\|_{C^1} < \varepsilon$ and $\|H(1,.) - f_2\|_{C^1} < \varepsilon$.

Proof of Lemma 4.3. Step 1 Let (f_1, f_2) in $\mathcal{S}_0(F)$, we define $\hat{h} : [0,1]\times M \to G(F)$ by

$$\hat{h}(t,x) = (x, tf_1(x) + (1-t)f_2(x))$$

.

For every $\varepsilon > 0$, the mapping $\hat{h}$: can be approximated by a smooth mapping $h^\varepsilon : [0,1] \times M \to G(F)$ satisfying $< \varepsilon$ which we additionnaly assumed to be transverse to $M\times\{0\}$ [see Transversality Theorem 2.1, p74, Hirsch].The mapping $h^\varepsilon : [0,1]$ can be written as

$$h^\varepsilon(t,x) = (h_1^\varepsilon(t,x), h_2^\varepsilon(t,x))$$

for some mappings $h_1^\varepsilon : [0,1] \times M \to M$ and $h_2^\varepsilon : [0,1] \times M \to V$ and one has

$$\forall (t,x) \in [0,1] \times M \; h_2^\varepsilon(t,x) \in \hat{F}(h_1^\varepsilon(t,x)).$$

If $\bar{H}$ is close enough to $\hat{H}'$, then $\bar{H}_1$ will be close enough to the mapping $p : [0,1] \times M \to M$, defined by $p(t,x) = x$, which is a C^∞ submersion. But $\mathrm{Sub}^\infty([0,1] \times M, M)$ is open in $C^\infty([0,1] \times M, M)$ (Hirsch p36) hence $\bar{H}_1$ will also be a submersion if the approximation is closed enough. Then (?????) there exists $\bar{H}_1^{-1}$ $[0,1] \times M \to [0,1] \times M$ close to Id such that $\bar{H}_1(\bar{H}_1^{-1}(t,x)) = x$. We then define $H : [0,1] \times M \to G(F)$ by

$$H(t,x) = (x, \bar{H}_2(\bar{H}_1^{-1}(t,x))).$$

. We remark that we have for every $(t,x) \in [0,1] \times M$, $H(t,x) \in G(F)$ because $\bar{H}_2(H_1^{-1}(t,x)) \in F(\bar{H}_1(\bar{H}_1^{-1}(t,x))) = F(t,x)$).

Now we can notice that $\bar{H}$ being close enough to $\hat{H}'$ then H will be closed to $\bar{H}$ which is transverse to $M \times \{0\}$ and so will be transverse to $M \times \{0\}$.[see p74, Hirsch]. Futhermore, H is an approximation of $\hat{H}$, and so $\tilde{H} = \bar{H}_2(\bar{H}_1^{-1}(t,x))$ is an approximation of H'. In particular $\tilde{H}(0,.)$ is an approximation of $H'(0,.) = f_2$ and $\tilde{H}(1,.)$ is an approximation of $H'(1,.) = f_1$.

So, the approximation beeing close enough then we can suppose 0 are regular values of $H_2'(0,.)$ and of $H_2'(1,.)$. Lemma 4.2 implies that the mapping $H'(0,.)$ and $H'(1,.)$ are transverse to $M \times \{0\}$. Besides, We remark that $H'_{/\partial[0,1]\times M}$ is defined by $H'_{/\partial[0,1]\times M}(0,x) = H'(0,x)$ and $H'_{/\partial[0,1]\times M}(1,x) = H'(1,x)$. Finally, $H'_{/\partial([0,1]\times M}$ is transverse to $M \times \{0\}$

□.

ANCIEN For every $\varepsilon > 0$, the mapping $\hat{h}$: can be approximated by a smooth mapping $h^\varepsilon : [0,1] \times M \to G(F)$ satisfying $< \varepsilon$ which we additionnaly assumed to be transverse to $M \times \{0\}$ [see Transversality Theorem 2.1, p74, Hirsch].The mapping $h^\varepsilon : [0,1]$ can be written as

$$h^\varepsilon(t,x) = (h_1^\varepsilon(t,x), h_2^\varepsilon(t,x))$$

for some mappings $h_1^\varepsilon : [0,1] \times M \to M$ and $h_2^\varepsilon : [0,1] \times M \to V$ and one has

$$\forall (t,x) \in [0,1] \times M \ h_2^\varepsilon(t,x) \in \hat{F}(h_1^\varepsilon(t,x)).$$

If $\bar{H}$ is close enough to $\hat{H}'$, then $\bar{H}_1$ will be close enough to the mapping $p : [0,1] \times M \to M$, defined by $p(t,x) = x$, which is a C^∞ submersion. But $\mathrm{Sub}^\infty([0,1] \times M, M)$ is open in $C^\infty([0,1] \times M, M)$ (Hirsch p36) hence $\bar{H}_1$ will also be a submersion if the approximation is closed enough. Then (?????) there exists $\bar{H}_1^{-1}$ $[0,1] \times M \to [0,1] \times M$ close to Id such that $\bar{H}_1(\bar{H}_1^{-1}(t,x)) = x$. We then define $H : [0,1] \times M \to G(F)$ by

$$H(t,x) = (x, \bar{H}_2(\bar{H}_1^{-1}(t,x))).$$

. We remark that we have for every $(t,x) \in [0,1] \times M$, $H(t,x) \in G(F)$ because $\bar{H}_2(H_1^{-1}(t,x)) \in F(\bar{H}_1(\bar{H}_1^{-1}(t,x))) = F(t,x)$).

Now we can notice that $\bar{H}$ being close enough to $\hat{H}'$ then H will be closed to $\bar{H}$ which is transverse to $M \times \{0\}$ and so will be transverse to $M \times \{0\}$.[see p74, Hirsch]. Futhermore, H is an approximation of $\hat{H}$, and so $\tilde{H} = \bar{H}_2(\bar{H}_1^{-1}(t,x))$ is an approximation of H'. In particular $\tilde{H}(0,.)$ is an approximation of $H'(0,.) = f_2$ and $\tilde{H}(1,.)$ is an approximation of $H'(1,.) = f_1$.

So, the approximation beeing close enough then we can suppose 0 are regular values of $H_2'(0,.)$ and of $H_2'(1,.)$. Lemma 4.2 implies that the mapping $H'(0,.)$ and $H'(1,.)$ are transverse to $M \times \{0\}$. Besides, We remark that $H'_{/\partial[0,1]\times M}$ is

defined by $H'_{/\partial[0,1]\times M}(0,x) = H'(0,x)$ and $H'_{/\partial[0,1]\times M}(1,x) = H'(1,x)$. Finally, $H'_{/\partial([0,1]\times M}$ is transverse to $M \times \{0\}$

□.

Proof of Theorem 4.1

Now, using the previous lemma, there exists H such that $\tilde{H}(0,.)$ is an approximation of f_2 and $\tilde{H}(1,.)$ is an approximation of f_1;

We can take the approximation close enough such that Lemma 1.2 implies

Card($f_1^{-1}(0)$) =Card($\tilde{H}(1,.)^{-1}(0)$)[2] and

Card($f_2^{-1}(0)$) =Card($\tilde{H}(0,.)^{-1}(0)$)[2].

Now, since $H : [0,1] \times M \to G(F)$ is a smooth mapping transverse to $M \times \{0\}$ which is a smooth manifold, we now (Hirsch, Theorem 33, p 22) that $H(M \times \{0\})$ is a smooth compact manifold whose dimension is $(m - (2m - (m+1)) = 1$. So $H^{-1}(M \times \{0\})$ is a finite union of closed-arcs whose boundary is empty, and of arcs whose boundaries are $\{a, b\}$ [Milnor...]

We know that $H_{/\partial([0,1]\times M)}$ is transverse to $M \times \{0\}$.

So, using Theorem 4.2 p31 (HIRSCH) we now that

$$H^{-1}_{/\partial([0,1]\times M}(M \times \{0\}) = \partial H^{-1}(M \times \{0\})$$

whose cardinal is even.

In conclusion, Card($H^{-1}_{/\partial[0,1]\times M}(M \times \{0\}$)=0 [2].

But Card($H^{-1}_{/\partial([0,1]\times M}(M \times \{0\}$)= Card($\tilde{H}^{-1}(1,.)(0)$)+Card($\tilde{H}^{-1}(0,.)(0)$)

=$Card(f_1^{-1}(0)$)+Card($f_2^{-1}(0)$)=0 [2]. Consequently, we have Card($f_1^{-1}(0)$) =Card($f_2^{-1}(0)$)[2].

□

A METTRE ICI LES LEMMES ET PEUT ETRE D'AUTRES PROPRIETES (produite et existence d'un zero lorsque 0 n'est pas forcement regulier)

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