

# Existence of financial equilibria : space of transfers of fixed dimension

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## Abstract

In this article, we consider the classical two-period exchange model with the same level of generality as the Arrow-Debreu model with non ordered preferences as in Gale and Mas Colell. So, the preferences of the consumers are not assumed to be complete or transitive and no assumption of monotonicity or free disposal is made. We present an existence theorem of pseudo-equilibria which implies the existence of financial equilibria under an assumption of regularity of the space of admissible transfers. In particular, we obtain the existence in the case of numeraire assets, of Arrow-Debreu contingent markets, thus generalizing the previous work by Duffie Shaffer.

**KEY WORDS:** existence of equilibria, financial equilibria, pseudo-equilibria, incomplete markets, real assets, fixed-point like theorem.

## Résumé

On considère dans cet article le modèle classique d'échange à deux-périodes similaire à celui d'Arrow-Debreu. En particulier, les préférences des consommateurs ne sont pas supposées complètes ou transitives comme dans Gale et Mas Colell (1975). De plus, aucune hypothèse de monotonie ou de libre disposition n'est faite. Utilisant un théorème de point fixe, on prouve l'existence de pseudo-équilibres, ce qui implique l'existence d'équilibres financiers sous une hypothèse de régularité de l'espace de transfert. On obtient en particulier l'existence d'équilibres dans le cadre nominal, dans le cas d'actifs contingents,... Ces résultats généralisent les travaux de Duffie-Shaffer (1985).

**MOTS CLÉS:** existence d'équilibres, équilibres financiers, pseudo-équilibres, marchés incomplets, actifs réels, théorème de point fixe.

# 1 Introduction

In this article, we consider the classical two-period exchange model with the same level of generality as the Arrow-Debreu model with non ordered preferences as in Gale and Mas Colell. So, the preferences of the consumers are not assumed to be complete or transitive and no assumption of monotonicity or free disposal is made. We present an existence theorem of pseudo-equilibria which implies the existence of financial equilibria under an assumption of regularity of the space of admissible transfers. In particular, we obtain the existence in the case of numeraire assets, of Arrow-Debreu contingent markets, thus generalizing the previous work by Duffie Shaffer.

The chapter is organized as follows. Section 2 quickly presents the model and recalls the definitions of financial equilibria. The main existence result is stated and some consequences are given to various economic models such as: nominal assets, numeraire assets, contingent commodities, pure spot markets. The proof of the main existence result is given in Section 3.

## 2 The existence result

### 2.1 The basic two period exchange economy

We consider the model of an exchange economy  $\mathcal{E}$  with a positive finite number  $m$  of consumers, two periods  $t = 0$  and  $t = 1$ , and a positive number  $K$  of divisible goods available at each period.

The uncertainty in period  $t = 1$  is represented by  $S$  states of nature. Only one state happens and it is only known at the beginning of the period. For convenience, the unique state of nature (known with certainty) today (i.e at  $t = 0$ ) will be denoted  $s = 0$ . Hence, the number of commodities available either at  $t = 0$  (with certainty) or at  $t = 1$  (contingent on each of the finite number  $S$  of possible states of nature) is  $K(1 + S)$ . We denote by  $X_i \subset R^{K(1+S)}$  the consumption set of the  $i$ -th consumer, by  $e_i \in R^{K(1+S)}$  her/his initial endowment vector. For all  $x = (x_i)_{i=1}^m \in \prod_{i=1}^m X_i$ , we let  $P_i(x) \subset X_i$  be the set of consumption plans which are strictly preferred to  $x_i$  by the consumer  $i$ , given the consumption plans  $(x_j)_{j \neq i}$  of the other consumers.

This general framework to describe the tastes of the consumers encompasses the case where the consumer  $i$  has a preference relation  $\preceq_i$  which is a binary relation on  $X_i$ . In this case, for all  $x = (x_i)_{i=1}^m \in \prod_{i=1}^m X_i$ ,  $P_i(x) = \{x'_i \in X_i \mid x_i \prec_i x'_i\}$  where the strict preference relation  $\prec_i$  is defined by  $x_i \prec_i x'_i$  if [ $x_i \preceq_i x'_i$  and not  $x'_i \preceq_i x_i$ ].

At each state  $s = 0, 1, \dots, S$ , there is a spot market for each of the  $K$  physical goods. In addition, we assume that there exist at time  $t = 0$  financial markets for a positive number  $J$  of assets. Given the price  $p = (p(0), \dots, p(S)) \in R^{K(1+S)}$

of the commodities, the asset  $j$  ( $j = 1, \dots, J$ ) can be bought at time  $t = 0$  and delivers at time  $t = 1$  a financial return  $W_{s,j}^1(p)$  (in unit of account) if state  $s$  prevails. Given the price vector  $p$ , a portfolio is a vector  $(z_1, \dots, z_J) \in R^J$  specifying the quantity  $z_j$  of each asset, with the usual convention that if  $z_j > 0$  then  $|z_j|$  represents the quantity of asset  $j$  bought at period 0 and if  $z_j < 0$  then  $|z_j|$  represents the quantity of asset  $j$  sold at period 0.

Given  $p \in R^{K(1+S)}$  the price vector of the commodities and given  $q \in R^J$  the price vector of the  $J$  assets, to buy  $z_j$  units of the  $j$ -th asset ( $j = 1, \dots, J$ ), that is, to buy the portfolio  $z = (z_j) \in R^J$ , consumer  $i$  has to pay  $q \cdot z = \sum_{j=1}^J q_j z_j$  at time  $t=0$  and he will get a return (in unit of account)  $\sum_{j=1}^J z_j W_{s,j}^1(p)$  at time  $t = 1$  if state  $s$  prevails. In the following, we denote by  $W^1(p)$  the  $S \times J$ -matrix of returns at time  $t = 1$ , that is,

$$W^1(p) = (W_{s,j}^1(p))_{s=1, \dots, S; j=1, \dots, J}$$

and  $W(p, q)$ , the  $(1 + S) \times J$ -matrix of returns across time  $t = 0$  and  $t = 1$  and states  $s = 1, \dots, S$  (or equivalently across states  $s = 0, \dots, S$ ) is defined by

$$W(p, q) = \begin{pmatrix} -q \\ W^1(p) \end{pmatrix}.$$

Hence, if  $z_i \in R^J$  denotes the portfolio of the  $i$ -th consumer, then  $W(p, q)z_i \in R^{1+S}$  is the vector of the financial returns, for consumer  $i$ , across the  $1 + S$  states of nature.

The economy  $\mathcal{E}$  can be summarized by the list

$$\mathcal{E} = ((X_i, P_i, e_i)_{i=1, \dots, m}, W^1).$$

## 2.2 Financial equilibria and no-arbitrage prices

We now formally define the notion of financial equilibria. In the following, if  $p \in R^{K(1+S)}$  [resp.  $\lambda \in R^{1+S}$ ] and  $x \in R^{K(1+S)}$ , we denote  $p \square x$  [resp.  $\lambda \square p$ ] the vector in  $R^{1+S}$  [resp.  $R^{K(1+S)}$ ] defined by

$$p \square x = (p(0) \cdot x(0), p(1) \cdot x(1), \dots, p(S) \cdot x(S))$$

$$[\text{resp. } \lambda \square p = (\lambda(0)p(0), \lambda(1)p(1), \dots, \lambda(S)p(S))]$$

where  $x \cdot y$  denotes the scalar product of two vectors  $x, y$  in  $R^K$ .

**Definition 2.1** *A financial equilibrium of the economy  $\mathcal{E}$  is an element*

*$((\bar{x}_i)_{i=1}^m, (\bar{z}_i)_{i=1}^m, \bar{p}, \bar{q})$  of  $\prod_{i=1}^m X_i \times R^{Jm} \times R^{K(1+S)} \times R^J$  such that, if we let  $\bar{x} = (\bar{x}_i)_{i=1}^m$ , one has:*

*(i) for every  $i = 1, \dots, m$ ,  $\bar{p} \square (\bar{x}_i - e_i) \leq W(\bar{p}, \bar{q})\bar{z}_i$  and  $P_i(\bar{x}) \cap B_i(\bar{p}, \bar{q}) = \emptyset$ , where*

$$B_i(\bar{p}, \bar{q}) = \{x_i \in X_i \mid \exists z_i \in R^J, \bar{p} \square (x_i - e_i) \leq W(\bar{p}, \bar{q})z_i\};$$

- (ii)  $\sum_{i=1}^m \bar{x}_i = \sum_{i=1}^m e_i$ ;  
 (iii)  $\sum_{i=1}^m \bar{z}_i = 0$ .

Let  $V$  be a Euclidean space and let  $r$  be a non-negative integer,  $r \leq \dim V$ . We denote  $G^r(V)$  the set consisting of all linear subspaces of  $V$  of dimension  $r$ , called the  $r$ -Grassmann manifold of  $V$ , and we let  $G(V) = \cup_{r=0}^{\dim V} G^r(V)$  called the Grassmann manifold of  $V$ . We now define the notion of no-arbitrage.

**Definition 2.2** We say that  $E \in G(R^{1+S})$  is a no-arbitrage space if  $E \cap R_+^{1+S} = \{0\}$ . We say that an  $(1+S) \times J$ -matrix  $W$  is a no-arbitrage matrix if  $\text{Im}W$  is a no-arbitrage space in  $R^{1+S}$ . We finally say that  $(p, q) \in R^{K(1+S)} \times R^J$  is a no-arbitrage price vector of the economy  $\mathcal{E} = ((X_i, P_i, e_i)_{i=1, \dots, m}, W^1)$  if the matrix

$$W := W(p, q) = \begin{pmatrix} -q \\ W^1(p) \end{pmatrix}.$$

is a no-arbitrage matrix.

At this stage, we notice that if  $E \in G(R^{1+S})$  is a no-arbitrage space then one has  $\dim E \leq S$ . We give below a well-known result, used in the following, and which gives a "dual property" of no-arbitrage.

**Lemma 2.1** Let  $W$  be an  $(1+S) \times J$ -matrix, then the two following assertions are equivalent:

- (\*)  $\text{Im}W \cap R_+^{1+S} = \{0\}$ ;  
 (\*\*)  $\exists \lambda \in R_+^{1+S}, \text{Im} W \subset \lambda^\perp$ .

**Proof of Lemma 2.1** Suppose that (\*) is true; we let  $\Delta = \{(x_1, \dots, x_{1+S}) \in R_+^{1+S} \mid \sum_{i=1}^{1+S} x_i = 1\}$  be the unit simplexe of  $R^{1+S}$ . From the no-arbitrage condition, we have  $\text{Im}W \cap \Delta = \emptyset$ . Since  $\Delta$  is compact and  $\text{Im}W$  is closed, the strict separation theorem implies there exist  $\lambda \in R^{1+S} \neq 0$  and  $\alpha \in R$  such that

$$\sup_{x \in \text{Im}W} \lambda \cdot x \leq \alpha < \inf_{y \in \Delta} \lambda \cdot y.$$

Since the linear mapping  $x \rightarrow \lambda \cdot x$  is majorized on the linear space  $\text{Im}W$ , it is equal to zero on  $\text{Im}W$ , which can be written  $\text{Im}W \subset \lambda^\perp$ . Now, it follows from the right inequality that all the components of  $\lambda$  are strictly positives, which proves (\*\*). The other implication is immediate.  $\square$

We now present the link between the notion of financial equilibria and the notion of no-arbitrage. For this we need first to introduce the following assumption:

**Non-Satiation Assumption (NS)** For every  $\bar{x} = (\bar{x}_i)_{i=1}^m \in \prod_{i=1}^m X_i$  such that  $\sum_{i=1}^m x_i = 0$ , for every  $i = 1, \dots, m$ , for every  $s_0 \in \{0, 1, \dots, S\}$ , there exists  $x_i \in X_i$  such that  $x_i(s) = \bar{x}_i(s)$  for every  $s \neq s_0$  and  $x_i \in P_i(\bar{x})$ .

**Proposition 2.1** *Let  $((\bar{x}_i)_{i=1}^m, (\bar{z}_i)_{i=1}^m, \bar{p}, \bar{q})$  be a financial equilibrium; then under the Non-Satiation Assumption (NS),  $(\bar{p}, \bar{q})$  is a no-arbitrage price.*

The proof of the above proposition is an immediate consequence of Proposition 2.2 and of Proposition 2.3 below.

## 2.3 Pseudo-equilibria

We now introduce a weaker notion of equilibria.

**Definition 2.3** *A pseudo-equilibrium of the economy  $\mathcal{E}$  is an element  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$  of  $\prod_{i=1}^m X_i \times R^{K(1+S)} \times R^J \times G(R^{1+S})$  such that, if we let  $\bar{x} = (\bar{x}_i)_{i=1}^m$ , one has:*

- (i) *for every  $i = 1, \dots, m$ ,  $\bar{x}_i \in B_i(\bar{p}, \bar{E})$  and  $P_i(\bar{x}) \cap B_i(\bar{p}, \bar{E}) = \emptyset$  where  $B_i(\bar{p}, \bar{E}) := \{x_i \in X_i \mid \exists t_i \in \bar{E}, \bar{p} \square (x_i - e_i) \leq t_i\}$ ;*
- (ii)  $\sum_{i=1}^m \bar{x}_i = \sum_{i=1}^m e_i$ ;
- (iii)  $\text{Im}W(\bar{p}, \bar{q}) := \{W(\bar{p}, \bar{q})z \mid z \in R^J\} \subset \bar{E}$ .

*Let  $r$  be a positive integer, we say that  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$  is a  $r$ -pseudo-equilibrium if it is a pseudo-equilibrium and  $\dim \bar{E} = r$ . The space  $\bar{E}$  associated to the pseudo-equilibrium is called its "transfer space".*

The link between the previous notion of pseudo-equilibrium and another one, commonly used in the literature, is given in the Appendix.

The first proposition states that, to every financial equilibrium, we can associate a pseudo-equilibrium as follows:

**Proposition 2.2** *Let  $((\bar{x}_i)_{i=1}^m, (\bar{z}_i)_{i=1}^m, \bar{p}, \bar{q})$  be a financial equilibrium, then*

*$((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \text{Im}W(\bar{p}, \bar{q}))$  is a pseudo-equilibrium.*

**Proof of Proposition 2.2.** Immediate from the definition.  $\square$

The following proposition shows that, under the Non-Satiation Assumption (NS), the transfer space of every pseudo-equilibrium is a no-arbitrage space. Formally,

**Proposition 2.3** *Let  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$  be a pseudo-equilibrium. Then, under the Non-satiation Assumption, the transfer-space  $\bar{E}$  is a no-arbitrage space.*

**Proof of Proposition 2.3.** Let  $\bar{x} = (x_i)_{i=1}^m$ ; suppose that  $\bar{E}$  is not a no-arbitrage space. Then there exists  $t = (t_s)_{s=1, \dots, S}$  in  $\bar{E}$  such that  $t(s) \geq 0$  for every  $s = 0, \dots, S$  and  $t_{s_0} > 0$  for some  $s_0$ ,  $0 \leq s_0 \leq S$ . Under the Non-satiation

Assumption for consumer 1, there exists  $x_1 \in P_1(\bar{x})$  such that  $x_1(s) = \bar{x}_1(s)$  for every  $s \neq s_0$ . But, from the condition (i) of pseudo-equilibrium, there exists  $\bar{t}_1 \in \bar{E}$  such that  $\bar{p} \square (\bar{x}_1 - e_1) \leq \bar{t}_1$ . Hence, for every integer  $n$  large enough,  $\bar{p} \square (x_1 - e_1) \leq nt + \bar{t}_1$ . This is a contradiction with the optimality of  $x_1$  in the budget set of consumer 1.  $\square$

**Remark.** From the above proposition, the transfer space  $\bar{E}$  of every pseudo-equilibrium satisfy  $\bar{E} \cap R_+^{1+S} = \{0\}$ . Hence,  $\bar{E}$  has a dimension less or equal to  $S$ , i.e,  $\dim \bar{E} \leq S$ .

The following proposition shows that to every pseudo-equilibrium  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$  of the economy  $\mathcal{E}$ , is associated a financial-equilibrium  $((\bar{x}_i)_{i=1}^m, (\bar{z}_i)_{i=1}^m, \bar{p}, \bar{q})$  if  $\bar{E} = \text{Im}W(\bar{p}, \bar{q})$ .

**Proposition 2.4** *Let  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E}) \in \prod_{i=1}^m X_i \times R^{K(1+S)} \times R^J \times G(R^{1+S})$  be a pseudo-equilibrium and assume that  $\bar{E} = \text{Im}W(\bar{p}, \bar{q})$ . Then there exist  $\bar{z}_i \in R^J$  ( $i = 1, \dots, m$ ) such that  $((\bar{x}_i)_{i=1}^m, (\bar{z}_i)_{i=1}^m, \bar{p}, \bar{q})$  is a financial equilibrium.*

**Proof of Proposition 2.4.** Let  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$  be a pseudo-equilibrium of  $\mathcal{E}$ . For every  $i = 1, \dots, m$ , there exists  $\bar{t}_i \in \bar{E}$  such that:

$$\bar{p} \square (\bar{x}_i - e_i) \leq \bar{t}_i.$$

Summing up over  $i$  these inequations, we obtain

$$\bar{p} \square \sum_{i=1}^m (\bar{x}_i - e_i) \leq \sum_{i=1}^m \bar{t}_i.$$

But condition (ii) of the definition of pseudo-equilibria gives  $\sum_{i=1}^m (\bar{x}_i - e_i) = 0$ .

Thus,  $\sum_{i=1}^m \bar{t}_i \geq 0$  and  $\sum_{i=1}^m \bar{t}_i \in \bar{E} = \text{Im}W(\bar{p}, \bar{q})$ . From Proposition 2.3,  $\bar{E} = \text{Im}W(\bar{p}, \bar{q})$  is a no-arbitrage space, hence  $\sum_{i=1}^m \bar{t}_i = 0$ . Finally,  $\sum_{i=1}^m (\bar{p} \square (\bar{x}_i - e_i) - \bar{t}_i) = 0$ , and each term of this sum belongs to  $-R_+^{1+S}$ . Hence, each of these terms is null, that is:

$$\bar{p} \square (\bar{x}_i - e_i) = \bar{t}_i, \text{ for every } i = 1, \dots, m.$$

Since  $\bar{t}_i \in \bar{E} = \text{Im}W(\bar{p}, \bar{q})$ , for every  $i = 1, \dots, m$ , there exists portfolios  $z_i \in R^J$  such that  $\bar{t}_i = W(\bar{p}, \bar{q})z_i$ . So we obtain

$$0 = \bar{p} \square \sum_{i=1}^m (\bar{x}_i - e_i) = W(\bar{p}, \bar{q}) \left( \sum_{i=1}^m z_i \right).$$

We now let  $\bar{z}_i = z_i - (1/m) \left( \sum_{i=1}^m z_i \right)$  for every  $i = 1, \dots, m$ . Then, the condition (iii) of the financial equilibrium holds:  $\sum_{i=1}^m \bar{z}_i = 0$ . Furthermore, we have

$$\bar{p} \square (\bar{x}_i - e_i) = \bar{t}_i = W(\bar{p}, \bar{q})z_i = W(\bar{p}, \bar{q})\bar{z}_i. \square$$

**Proposition 2.5** *Let  $1 = (1, \dots, 1)$  and  $\lambda \in R_{++}^{1+S}$ . If  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$  is a pseudo-equilibrium of  $\mathcal{E}$  such that  $\bar{E} \subset 1^\perp$ , then  $((\bar{x}_i)_{i=1}^m, \bar{p}', \bar{q}', \bar{E}')$  is a pseudo-equilibrium of  $\mathcal{E}$  such that  $\bar{E}' \subset \lambda^\perp$ , where  $p' \in R^{K(1+S)}$  is defined by  $p'(s) =$*

$p(s)/\lambda(s)$  for  $s = 0, \dots, S$ , where  $\bar{E}' = \{(y(0)/\lambda(0), y(1)/\lambda(1), \dots, y(S)/\lambda(S) \mid (y(0), y(1), \dots, y(S)) \in \bar{E}\}$  and where  $\bar{q}'_j = -(1/\lambda(0)) \sum_{s=1}^S \lambda(s) W_{s,j}^1(\bar{p}')$  for every  $j = 1, \dots, J$ .

**Proof of Proposition 2.5** The proof is immediate.  $\square$

## 2.4 Existence of pseudo-equilibria and equilibria

We first set the following assumptions which describe the general framework of the paper.

**Consumption Assumption (C)** For every  $i = 1, \dots, m$ ,  $X_i$  is closed in  $R^{K(1+S)}$ , convex and bounded below; the correspondence  $P_i$  is lower semicontinuous<sup>1</sup> with values which are convex and open in  $X_i$  (for its relative topology).

**Strong Survival Assumption (SS)** For every  $i = 1, \dots, m$ ,  $e_i \in \text{int}X_i$ .

We now denote by  $\mathcal{A}(\mathcal{E})$  the set of attainable allocations of the economy, that is,

$$\mathcal{A}(\mathcal{E}) = \{(x_1, \dots, x_m) \in X_1 \times \dots \times X_m \mid \sum_{i=1}^m x_i = e\}.$$

**Non-Satiation Assumption (NS)** For every  $\bar{x} = (\bar{x}_i)_{i=1}^m \in \mathcal{A}(\mathcal{E})$ , for every  $i = 1, \dots, m$ , for every  $s_0 \in \{0, 1, \dots, S\}$ , there exists  $x_i \in X_i$  such that  $x_i(s) = \bar{x}_i(s)$  for every  $s \neq s_0$  and  $x_i \in P_i(\bar{x})$ .

**Asset Assumption (AS)** The  $S \times J$  return matrix at time  $t = 1$  is a continuous mapping in the price  $p$ , that is, the mapping  $p \rightarrow W^1(p)$  is continuous on  $R^{K(1+S)}$ .

**Definition 2.4** We let  $r(W^1)$  be the smallest non-negative integer  $r$  such that there exist  $r$  continuous mappings  $a_i(\cdot)$ ,  $i = 1, \dots, r$ , from  $R^{K(1+S)}$  to  $R^{1+S}$  such that for every  $p \in R^{K(1+S)}$ ,  $\text{Im}W^1(p) \subset \text{span}\{a_1(p), \dots, a_r(p)\}$ .

**Remark 1** We notice that  $r(W^1) \leq \min\{J, S\}$ . Furthermore, when  $W^1$  does not depend upon the price  $p$  (as in the nominal case), then  $r(W^1) = \text{rank}W^1$ .

We now state the first existence theorem the proof of which is given in the next section:

**Theorem 2.1** Under Assumptions (C), (SS), (NS), (AS), for every integer  $r$  such that  $S \geq r \geq r(W^1)$ , for every  $\lambda \in R_{++}^{1+S}$ , there exists a  $r$ -pseudo-equilibrium  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$  of  $\mathcal{E}$ , such that  $\bar{E} \subset \lambda^\perp$ ;

<sup>1</sup>The correspondence  $P_i$  is said to be upper semicontinuous (u.s.c.), [resp. lower semicontinuous (l.s.c.)] if the set  $\{x \in X \mid P_i(x) \subset U\}$  [resp.  $\{x \in X \mid P_i(x) \cap U \neq \emptyset\}$ ] is open in  $X$  for every open set  $U \subset Y$ .

**Proof of Theorem 2.1** The proof of the existence of a pseudo-equilibrium  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$  will be given in section 3, in the case where  $\lambda = (1, 1, \dots, 1)$ ; then the existence of pseudo-equilibria in the more general case where  $\lambda \in R_{++}^{1+S}$  will be a consequence of Proposition 2.5  $\square$

**Remark 2** We can notice that the condition (iii) of the definition of pseudo-equilibria and the condition  $\bar{E} \subset \lambda^\perp$  implies that  $\bar{q}_j = -(1/\lambda(0)) \sum_{s=1}^S \lambda(s) W_{s,j}^1(\bar{p})$  for every  $j = 1, \dots, J$ .

**Remark 3** Let  $r_1(W^1) = \min_{p \in B(0,1)} \text{rank } W^1(p)$ . We have  $r_1(W^1) \leq r(W^1)$ ; if  $r(W^1) = r^1(W^1)$  every  $r$ -pseudo-equilibrium is a financial equilibrium and we know that it can fail to exist (cf. Hart (1975) counterexample). But it is an open question whether for  $r(W^1) > r > r^1(W^1)$  there would exist  $r$ -pseudo-equilibria.

When we assume that  $J \leq S$ , as it is done traditionally in the literature, we have the following result:

**Corollary 2.1** *Assume that  $J \leq S$  and that the assumptions (C), (SS), (NS), (AS) hold. For every  $\lambda \in R_{++}^{1+S}$ , there exists a  $J$ -pseudo-equilibrium  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$  of  $\mathcal{E}$ , such that  $\bar{E} \subset \lambda^\perp$ .*

**Proof of Corollary 2.1** We have  $S \geq J \geq r(W^1)$ , and we apply Theorem 2.1.  $\square$

We now state the second existence theorem on financial equilibrium.

**Theorem 2.2** *Let  $\Lambda \in R_{++}^{1+S}$ . If the exchange economy  $\mathcal{E} = ((X_i, P_i, e_i)_{i=1, \dots, m}, W^1(\cdot))$  satisfies the Assumptions (C), (SS), (NS), (AS) and if  $r^1(W^1) = r(W^1)$  then there exists a financial equilibrium  $((\bar{x}_i)_{i=1}^m, (\bar{z}_i)_{i=1}^m, \bar{p}, \bar{q})$  of  $\mathcal{E}$  such that  $\text{Im}W \subset \lambda^\perp$ .*

**Proof of Theorem 2.2** It is an immediate consequence of Theorem 2.1 and of Proposition 2.4.  $\square$

**Remark 4** The condition  $r^1(W^1) = r(W^1)$  is stronger than the condition "rank  $W^1(p)$  is constant for every  $p$ ". It is an open question to know whether there exist equilibria under this latest assumption.

**Remark 5** If the rank of  $W^1(p)$  is constant, equal to  $J$ , then we obtain the condition  $r^1(W^1) = r(W^1)$ , and so the existence of equilibria.

#### 2.4.1 Fixed space of return: the nominal case

When  $W^1(p)$ , the matrix of return at time  $t = 1$ , does not depend upon the price  $p$ , the financial structure is called *nominal* (or *purely financial*) *asset structure* and we shall simply denote this  $S \times J$ -matrix  $W^1$ .

The existence of financial equilibria in the case of nominal assets is given by the next theorem, which also characterizes equilibria asset prices to be any no-arbitrage asset price.

We first say that  $q \in R^J$  is a no-arbitrage asset price if the matrix:

$$W = \begin{pmatrix} -q \\ W^1 \end{pmatrix}$$

is a no-arbitrage matrix.

**Theorem 2.3** Consider an exchange economy  $\mathcal{E} = ((X_i, P_i, e_i)_{i=1, \dots, m}, W^1)$  whose financial structure is defined by nominal assets, that is,  $W^1$  does not depend upon the price  $p$  and assume that  $\mathcal{E}$  satisfies the assumptions (C), (SS) and (NS). Let  $q \in R^J$ , then the two following assertions are equivalent:

- (i)  $q$  is a non-arbitrage asset price;
- (ii) there exists a financial equilibrium  $((\bar{x}_i)_{i=1}^m, (\bar{z}_i)_{i=1}^m, \bar{p}, \bar{q})$  of  $\mathcal{E}$  such that  $q = \bar{q}$ .

**Proof of the Theorem 2.3.** (ii) implies (i) is an immediate consequence of Proposition 2.1. Now suppose that (i) is true. Let  $\lambda \in R_{++}^{1+S}$  such that  ${}^t\lambda \cdot \text{Im} \begin{pmatrix} -q \\ W^1 \end{pmatrix} = 0$ . We now apply Theorem 2.2: there exists a financial equilibrium  $((\bar{x}_i)_{i=1}^m, (\bar{z}_i)_{i=1}^m, \bar{p}, \bar{q})$  of  $\mathcal{E}$  such that  $\bar{E} \subset \lambda^\perp$  which implies  $q = \bar{q}$ .  $\square$

**Remark** An important case of nominal assets is the case where no financial transactions are possible, that is the case of pure spot markets. In this case, the matrix of return  $W^1$  is the nul-matrix.

## 2.4.2 rank of the return-matrix fixed

*the contingent goods case*

We now deduce an existence theorem for a financial structure defined by contingent good as in the standart Arrow-Debreu model. A contingent good  $j = (s_0, k_0)$  delivers 1 unit of commodity  $k_0$  if state  $s_0$  prevails. Formally, if  $j = (s_0, k_0)$ ,

$$W_{s,j}^1(p) = 0 \text{ if } s \neq s_0$$

$$W_{s_0,j}^1(p) = p_k(s_0).$$

**Corollary 2.2** If the exchange economy  $\mathcal{E}_{AD} = ((X_i, P_i, e_i)_{i=1, \dots, m}, W^1)$  satisfies assumptions (C), (SS) and the following one

**Monotonicity Assumption** For every  $x \in \mathcal{A}(\mathcal{E})$ , for every  $i$ , for every  $s$ , defining  $e_s \in R^{K(1+S)}$  by  $e_s(s) = e$  and  $e_s(s') = 0$  if  $s' \neq s$ ,  $x^i + e_s \in P_i(x_i)$

then  $\mathcal{E}_{AD}$  has a financial equilibrium such that  $\text{Im}W(p) \subset (1, 1, \dots, 1)^\perp$  and  $(p(0), p(1), \dots, p(s))$  is an AD price equilibrium.

### Proof of Corollary 2.2

Take the  $KS \times S$  matrix  $R = \begin{pmatrix} Id_{S \times S} \\ 0 \end{pmatrix}$ . The matrix  $W_1(p)R = \text{diag}(p_1(1), \dots, p_1(S))$  is a diagonal with strictly positive element under monotonicity assumption. So  $r^1(W^1) = r(W^1)$  and we apply Corollary 2.2. Let  $((\bar{x}_i)_{i=1}^m, (\bar{z}_i)_{i=1}^m, \bar{p}, \bar{q})$  be an equilibrium such that  $\text{Im}W(p) \subset (1, 1, \dots, 1)^\perp$ ; taking the scalar product of each side of budget-equations  $\bar{p} \square (x_i - e_i) \leq W(\bar{p}, \bar{q})z_i$  with the vector  $(1, 1, \dots, 1)$ , we obtain:

$$\bar{p} \cdot (\bar{x}_i - e_i) \leq 0, \text{ for every } i = 1, \dots, m. \quad (1)$$

Besides, since  $B_i(\bar{p}, \bar{E}) \subset B_i^{AD}(\bar{p})$ , where  $B_i^{AD}(\bar{p})$  is the arrow-debreu budget associated to price  $\bar{p}$ , we obtain  $P_i(\bar{x}) \cap B_i^{AD}(\bar{p}) = \emptyset$  for every  $i = 1, \dots, m$  which ends the proof.  $\square$

*the numeraire case*

A *numeraire assets* is defined as follows. Let us choose a consumption bundle  $e \in R^K$  as a unit of “numeraire” (instead of a single good), a numeraire assets is a real assets which delivers the bundle  $a_j(s) = r_j(s)e$  where  $r_j(s) \in R$  denotes the random return of asset  $j$  across the states  $s$  of the world at  $t = 1$ . If all assets are numeraire assets,  $e$  and the  $(S \times J)$ -matrix  $R = (r_j(s))_{\substack{s=1, \dots, S \\ j=1, \dots, J}}$  summarize the *numeraire asset structure* and

$$W_{s,j}^1(p) = ((p(s) \cdot e)r_j(s)).$$

Existence of financial equilibria in the case of numeraire assets is given by the following corollary:

**Corollary 2.3** *Consider an exchange economy  $\mathcal{E} = ((X_i, Z_i, P_i, e_i)_{i=1, \dots, m}, W^1)$  whose financial structure is defined by numeraire assets, that is,*

$$W_{s,j}^1(p) = ((p(s) \cdot e)r_j(s))$$

*and assume that the matrix  $R = (r_j(s))_{\substack{s=1, \dots, S \\ j=1, \dots, J}}$  has rank  $J$  and that the economy  $\mathcal{E}$  satisfies assumptions (C), (SS) and the following one*

**Monotonicity Assumption** *For every  $x \in \mathcal{A}(\mathcal{E})$ , for every  $i$ , for every  $s$ , defining  $e_s \in R^{K(1+S)}$  by  $e_s(s) = e$  and  $e_s(s') = 0$  if  $s' \neq s$ ,  $x^i + e_s \in P_i(x_i)$*

*then the model with numeraire assets has a financial equilibrium.*

**Proof of Corollary 2.4** It is easy to see that  $r^1(W^1) = r(W^1) = S$ , and we apply Corollary 2.2.

## 3 Proof of Theorem 2.1

The proof will consist in three steps. Firstly, we will suppose that  $r = J \leq S$ . Then, following Gale and Mas Colell (1975), we enlarge the preferred sets  $P_i(x)$

of the consumers and we bound their consumption sets in a classical way. This leads us to consider a new economy  $\hat{\mathcal{E}}^B$ . Then we show that every  $r$ -pseudo-equilibrium of  $\hat{\mathcal{E}}^B$  is an  $r$ -pseudo-equilibrium of the original economy  $\mathcal{E}$ .<sup>2</sup>

Secondly, we prove the existence of  $r$ -pseudo-equilibria of the economy  $\hat{\mathcal{E}}^B$ , as a consequence of a general existence result for economies with bounded consumption sets. We shall use a fixed-point argument in which we introduce an additional variable, the subspace  $E \in G^J(R^{1+S})$  as in previous work on the subject by Duffie-Schaffer.

We finally prove in the last section that the assumption  $r = J \leq S$  can be replaced by  $S \geq r \geq r(W^1)$ .

### 3.1 Modifying the economy $\mathcal{E}$

We now suppose that  $r = J \leq S$ .

#### 3.1.1 Definition of augmented preference

Following Gale and Mas-Colell (1975-1979), for  $x = (x_1, \dots, x_m) \in X_1 \times \dots \times X_m$ , we define the ‘‘augmented preferences’’  $\hat{P}_i$  by:

$$\hat{P}_i(x) = \cup_{x'_i \in P_i(x)} [x_i, x'_i] = \{x_i + \alpha(x'_i - x_i) \mid 0 < \alpha \leq 1, x'_i \in P_i(x)\} \subset X_i,$$

and we notice that we have  $P_i(x) \subset \hat{P}_i(x)$ .

We now define a new economy  $\hat{\mathcal{E}}$  which only differs from the original one  $\mathcal{E}$  by the fact that the original preferred sets  $P_i(x)$  are replaced by the larger ones  $\hat{P}_i(x)$  defined above. To summarize, we let

$$\hat{\mathcal{E}} := ((X_i, \hat{P}_i, e_i)_{i=1, \dots, m}, W).$$

The interest to consider the economy  $\hat{\mathcal{E}}$ , instead of  $\mathcal{E}$ , is twofold. Firstly,  $\hat{\mathcal{E}}$  satisfies more properties than  $\mathcal{E}$ , as shown in the following Proposition 3.1. Secondly, every pseudo-equilibrium of  $\hat{\mathcal{E}}$  is a pseudo-equilibrium of  $\mathcal{E}$ , as shown in Proposition 3.3 below.

**Proposition 3.1** (a) *If  $P_i$  is lower semicontinuous, then  $\hat{P}_i(x)$  is also lower semicontinuous; Let  $x \in \prod_{i=1}^m X_i$ , then*

- (b) *If  $P_i(x)$  is convex, then  $\hat{P}_i(x)$  is also convex;*
- (c) *If  $P_i(x)$  is open in  $X_i$  then  $\hat{P}_i(x)$  is also open in  $X_i$ ;*
- (d) *for every  $x_i \in \hat{P}_i(\bar{x})$  then  $[x_i, \bar{x}_i] \in \hat{P}_i(\bar{x})$ .*

---

<sup>2</sup>We shall show in Annex 2 that in fact the economies  $\mathcal{E}$  and  $\hat{\mathcal{E}}^B$  have the same sets of equilibria, a result which we do not need at this stage, but which is also of interest for itself.

**Proof of (a)** Suppose that  $P_i$  is lower semi-continuous. Let  $(x)_{i=1}^m \in X = \prod_{i=1}^m X_i$ ,  $y \in \hat{P}_i(x)$  and  $(x^n)$  converging to  $x$ . Then there exists  $z \in P_i(x)$  and  $\lambda \in ]0, 1[$  such that  $y = \lambda z + (1 - \lambda)x_i$ . Since  $P_i$  is lower semi-continuous, there exists  $z^n \in P_i(x^n)$  converging to  $z$ . Then if we let  $y^n = \lambda z^n + (1 - \lambda)x_i^n$  we have  $y^n \in \hat{P}_i(x^n)$  and  $y^n$  converges to  $y$ , which proves that  $\hat{P}_i$  is lower semi-continuous.  $\square$

**Proof of (b).** Suppose that  $P_i(x)$  is convex. Let  $(x', x'')$  be two elements of  $\hat{P}_i(x)$ . By definition, it exist  $\lambda$  and  $\lambda'$  in  $[0, 1[$  such that  $x' = \lambda x + (1 - \lambda)a$ ,  $x'' = \lambda' x + (1 - \lambda')a'$ ,  $a \in P_i(x)$  and  $a' \in P_i(x)$ . We want to show that for every  $\mu \in [0, 1]$  we have :  $\mu x' + (1 - \mu)x'' \in \hat{P}_i(x)$ .

But  $\mu x' + (1 - \mu)x'' = (1 - \nu)x + (\nu)a''$  where  $a'' = (\mu\lambda)/\nu.a + \lambda'(1 - \mu)/\nu.a'$  and  $\nu = (\mu\lambda + (1 - \mu)\lambda')$ . Since  $P_i(x)$  is convex, we have  $a'' \in P_i(x)$ . Finally, since  $\nu \in [0, 1[$ , we have  $\mu x' + (1 - \mu)x'' \in \hat{P}_i(x)$  and so  $\hat{P}_i(x)$  is convex.  $\square$

**Proof of (c)** Suppose that  $P_i(x)$  is open in  $X_i$ . Let  $x \in X$ ,  $y \in \hat{P}_i(x)$ . Then there exists  $z \in P_i(x)$  and  $\lambda \in ]0, 1[$  such that  $y = \lambda z + (1 - \lambda)x_i$ . Since  $P_i(x)$  is open, there exists  $\epsilon$  such that if  $z' \in B(z, \epsilon)$  then  $z' \in P_i(x)$ . Let  $z' \in P_i(x)$  and  $y' = \lambda z' + (1 - \lambda)x_i$ . Then  $y - y' = \lambda(z - z')$ . So, there exists  $\epsilon' < \epsilon$  such that if  $\|y - y'\| < \epsilon'$  then  $\|z - z'\| < \epsilon$ . Then if  $y' \in B(y, \epsilon')$  we have  $y' \in \hat{P}_i(x)$  which implies that  $\hat{P}_i(x)$  is open.  $\square$

**Proof of (d).** The Assumption (P) is an easy consequence of the definition of  $\hat{P}_i$ .  $\square$

### 3.1.2 Compactification of the economy

It follows from the Consumption Assumption (C) that the attainable set  $\mathcal{A}(\mathcal{E})$  is compact. Hence if we note  $\hat{X}_i$  the projection of  $\mathcal{A}(\mathcal{E})$  on  $X_i$ , the set  $\hat{X}_i$  is bounded, for every  $i = 1, \dots, m$ . Consequently, one can choose  $r > 0$  large enough such that

$$\hat{X}_i \subset \text{int}\bar{B}(0, r), \text{ for every } i = 1, \dots, m.$$

We let for every  $i = 1, \dots, m$ ,

$$X_i^B = X_i \cap \bar{B}(0, r),$$

and

$$\hat{P}_i^B(x) = \hat{P}_i(x) \cap X_i^B,$$

and we define a new economy  $\hat{\mathcal{E}}^B$  which only differs from  $\mathcal{E}$  by the fact the consumption sets  $X_i$  have been replaced by the above sets  $X_i^B$  and the correspondences  $P_i$  by  $\hat{P}_i^B$ . To summarize, we let

$$\hat{\mathcal{E}}^B := ((X_i^B, \hat{P}_i^B, e_i)_{i=1, \dots, m}, W).$$

**Proposition 3.2** *Under the Assumptions of Theorem 2.1, the characteristics of the economy  $\hat{\mathcal{E}}^B$  satisfy the following properties :*

(a)  $\mathcal{A}(\mathcal{E}) = \mathcal{A}(\hat{\mathcal{E}}^B)$ ;

(b) for every  $i = 1, \dots, m$ ,  $X_i^B$  is convex and compact, the correspondence  $\hat{P}_i^B$  is lower semicontinuous with values which are convex and open in  $X_i^B$  and for every  $\bar{x} = (\bar{x}_i)_{i=1}^m \in \prod_{i=1}^m X_i^B$ ,

for every  $x_i \in \hat{P}_i^B(\bar{x})$  then  $[x_i, \bar{x}_i] \in \hat{P}_i^B(\bar{x})$ ;

(c) for every  $i = 1, \dots, m$ ,  $e_i \in \text{int}X_i^B$ ;

(d) For every  $\bar{x} = (\bar{x}_i)_{i=1}^m \in \mathcal{A}(\hat{\mathcal{E}}^B)$ , for every  $i = 1, \dots, m$ , for every  $s = 0, 1, \dots, S$ , there exists  $x_i \in X_i^B$  such that  $x_i(s') = \bar{x}_i(s')$  for every  $s' \neq s$  and  $x_i \in \hat{P}_i^B(\bar{x})$ .

**Proof of Proposition 3.3.** left to the reader  $\square$

### 3.1.3 Equilibria of $\hat{\mathcal{E}}^B$ are equilibria of $\mathcal{E}$

This is a consequence of the next proposition.

**Proposition 3.3** Under Assumptions (C), (SS), (NS), let  $((\bar{x}_i)_{i=1}^m, \bar{E}, \bar{p}, \bar{q})$  be a pseudo-equilibrium (resp.  $((\bar{x}_i)_{i=1}^m, (\bar{z}_i)_{i=1}^m, \bar{p}, \bar{q})$  a financial equilibrium) of  $\hat{\mathcal{E}}^B$ , then it is also a pseudo-equilibrium (resp. a financial equilibrium) of  $\mathcal{E}$ .

**Proof of Proposition 3.3.** Let  $((\bar{x}_i)_{i=1}^m, (\bar{z}_i)_{i=1}^m, \bar{p}, \bar{q})$  be a financial equilibrium of  $\hat{\mathcal{E}}^B$ . In view of the definition of financial equilibrium, to prove that is also an equilibrium of  $\mathcal{E}$  we only have to prove that  $P_i(\bar{x}) \cap B_i(\bar{p}, \bar{q}) = \emptyset$  for every  $i = 1, \dots, m$ . Assume, that for some  $i$ , it is no true. We let  $x_i \in P_i(\bar{x}) \cap B_i(\bar{p}, \bar{q})$ . The condition (ii) of pseudo-equilibrium and the definition of  $r$  implies that  $x_i \in \text{int}\bar{B}(0, r)$ . The condition (i) of pseudo-equilibrium implies that  $\bar{x}_i \in B_i(\bar{p}, \bar{q})$ . It follows from the definition of  $\hat{P}$  that for some  $\alpha > 0$ ,  $\bar{x}_i + \alpha(x_i - \bar{x}_i) \in B_i^B(\bar{p}, \bar{q}) \cap P_i(\bar{x})$ , a contradiction. The proof is similar for financial equilibria.  $\square$

## 3.2 Existence of equilibria of $\hat{\mathcal{E}}^B$

In view of Proposition 3.2, if the economy  $\mathcal{E}$  satisfies the assumptions of theorem 2.1 then the economy  $\hat{\mathcal{E}}^B$  satisfies the assumptions of the following theorem which directly gives the existence of equilibria of  $\hat{\mathcal{E}}^B$ .

Without any risk of confusion, in the next theorem and the whole section, we denote the economy that we shall consider by  $\mathcal{E}$ .

We posit the following assumptions

**Assumption ( $\hat{C}$ )** For every  $i = 1, \dots, m$ , (a)  $X_i$  is **compact**, and convex;

(b) the correspondence  $P_i$  is lower semicontinuous with values which are convex and open in  $X_i$  (for its relative topology) and satisfies assertion (a), (b), (c), and (d) of Proposition 3.1;

(c) for every  $\bar{x} = (\bar{x}_i)_{i=1}^m \in \prod_{i=1}^m X_i$ , for every  $x_i \in P_i(\bar{x})$  then  $[x_i, \bar{x}_i] \in P_i(\bar{x})$ .

**Theorem 3.1** Under Assumptions  $(\hat{C})$ ,  $(SS)$ ,  $(NS)$  and  $(AS)$ , for every  $\lambda \in R_{++}^{1+S}$ , there exists a pseudo-equilibrium  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$  of  $\mathcal{E}$ , such that  $\bar{E} \subset \lambda^\perp$ .

The proof of Theorem 3.1 is given in the next sections.

### 3.2.1 Definition of correspondences

Let  $B$  be the closed unit ball of  $R^{K(1+S)}$  and let  $1$  be the vector in  $R^{1+S}$  whose coordinates are all equal to 1.

We now introduce some definitions in which  $((x_i)_{i=1}^m, p, E)$  is given in  $\prod_{i=1}^m X_i \times B \times G^J(R^{1+S})$  and in which we let  $x = (x_i)_{i=1}^m$ :

for  $i = 2, \dots, m$ , we consider the “augmented” budget set:

$$\begin{aligned}\beta_i(p, E) &= \{x_i \in X_i \mid \exists t_i \in E, p \square (x_i - e_i) \leq t_i + (1 - \|p\|) \cdot 1\}, \\ \alpha_i(p, E) &= \{x_i \in X_i \mid \exists t_i \in E, p \square (x_i - e_i) < t_i + (1 - \|p\|) \cdot 1\};\end{aligned}$$

for  $i = 1$ , following the so-called “Cass trick”, we consider the “augmented” “Arrow-Debreu” budget set:

$$\begin{aligned}\beta_1^{AD}(p, E) &= \{x_1 \in X_1 \mid p \cdot (x_1 - e_1) \leq 1 - \|p\|\}, \\ \alpha_1^{AD}(p, E) &= \{x_1 \in X_1 \mid p \cdot (x_1 - e_1) < 1 - \|p\|\};\end{aligned}$$

for  $i = 1, \dots, m$ , following Gale and Mas Colell (1975), we let:

$$\Phi_i(x, p, E) = \begin{cases} \{e_i\} & \text{if } x_i \notin \beta_i(p, E) \text{ and } \alpha_i(p, E) = \emptyset, \\ \beta_i(p, E) & \text{if } x_i \notin \beta_i(p, E) \text{ and } \alpha_i(p, E) \neq \emptyset, \\ \alpha_i(p, E) \cap P_i(x) & \text{if } x_i \in \beta_i(p, E); \end{cases}$$

for  $i = m + 1$ , the revision of prices is done according to the standard rule :

$$\Phi_{m+1}(x, p, E) = \{p' \in B \mid p' \cdot \sum_{i=1}^m (x_i - e_i) > p \cdot \sum_{i=1}^m (x_i - e_i)\};$$

and for  $j = 1, \dots, J$ ,

$$\psi_j(x, p, E) = \left(-\sum_{s=1}^S W_{s,j}^1(p), W_{1,j}^1(p), \dots, W_{S,j}^1(p)\right).$$

The properties of the above correspondences and mappings are summarized in the following lemma.

**Lemma 3.1** *For every  $i = 1, \dots, m + 1$ , the correspondence  $\Phi_i$  is lower semi-continuous and has convex (possibly empty) values. For every  $j = 1, \dots, J$ ,  $\psi_j$  is a continuous mapping.*

### Proof of Lemma 3.1

(a) We first remark that  $\alpha_i$  is an open graph correspondence.

(b) First, the survival assumption implies that:

$$e_i \in \beta_i(p, q) \text{ for every } p, q \in B$$

and one has

$$\alpha_i(p, q) \subset \gamma_i(p, q) \subset \beta_i(p, q) \text{ for every } p, q \in B$$

(c) We now define  $A_i = \{(p, E) \in B \times G^J(R^{1+S}) \mid \alpha_i(p, q) \neq \emptyset \text{ and } x_i \notin \beta_i(p, E)\}$ ,  $A'_i = \{(p, E) \in B \times G^J(R^{1+S}) \mid \alpha_i(p, q) = \emptyset \text{ and } x_i \notin \beta_i(p, E)\}$  and  $A''_i = \{(p, E) \in B \times G^J(R^{1+S}) \mid x_i \in \beta_i(p, E)\}$ .

We will prove that  $\beta_i$  is l.s.c on  $A_i$ . Let  $(x, p, E) \in O_{\bar{y}}$ ; We have  $\alpha_i(p, E) \neq \emptyset$ . Let  $x'_i \in X_i$  such that  $x'_i \in \beta_i(p, E)$ . Since  $\alpha_i(p, E)$  is nonempty, there exists  $y_i$  and  $t_i$  verifying  $p \square (y_i - e_i) \ll t_i + (1 - \|p\|)(1, \dots, 1)$ . Let  $(p^n, E^n)$  be a sequence converging to  $(p, E)$ . For  $n$  large enough, one has  $p^n \cdot (y_i - e_i) \ll t_i + (1 - \|p^n\|)(1, \dots, 1)$ . Let  $x_i^n = (1 - 1/n)x'_i + y_i/n$ . Then  $x_i^n \in \beta_i(p^n, E^n)$  and  $(x_i^n)$  converge to  $x$ . So, the restriction of  $\beta_i$  on  $A_i$  is lower semi-continuous.

(d) Now, we prove that  $\Phi_i$  is lower semi-continuous. It is clear that  $\Phi_i$  is lower semi-continuous on the set  $A_i$  since  $\Phi_i = \beta_i$  (by definition) on  $A_i$  and since we have shown above that  $A_i$  is open, and that  $\beta_i$  is lower semi continuous on the set  $A_i$ . We now prove that it is also lower semi continuous at the point  $(\bar{x}, \bar{p}, \bar{E}) \notin A_i$ . Suppose that there exist  $\bar{y} = (\bar{x}, \bar{p}, \bar{E}) \in \prod_{i=1}^m B \times G^J(R^{1+S})$  and an open subset  $U$  of  $X_i$  such that  $\Phi_i(\bar{y}) \cap U$  is not empty. Then let  $\bar{z}$  in  $\Phi_i(\bar{y}) \cap U$ . We will now distinguish two cases :

i) If  $\bar{y} \in A'_i$  i.e  $\bar{x}_i \notin \beta_i(\bar{p}, \bar{E})$  and  $\alpha_i(\bar{p}, \bar{E}) = \emptyset$ . Then one has  $\Phi_i(\bar{y}) = \{e_i\}$  and so  $\Phi_i(\bar{y}) \cap U = \{e_i\}$ . Since the set  $\{(x_i, p, E) \notin \beta_i(p, E)\}$  is an open subset of  $X_i \times B \times G^J(R^{1+S})$ , there exist an open neighborhood of  $\bar{y}$  called  $O_{\bar{y}}$  such that if  $(x, p, E) \in O_{\bar{y}}$  then  $x_i \notin \beta_i(p, E)$ . Now, let  $y = (x, p, E) \in O_{\bar{y}}$ . If  $\alpha_i(p, E) = \emptyset$  then  $\Phi_i(y) = \{e_i\}$  and so  $\Phi_i(y) \cap U = \{e_i\}$  is not empty. If  $\alpha_i(p, E) \neq \emptyset$  then  $\Phi_i(y) = \beta_i(p, E)$ . But the Survivance Assumption implies that  $e_i \in \beta_i(p, E)$  and so  $\Phi_i(y) \cap U = \{e_i\}$  is not empty.

ii) If  $\bar{y} \in A''_i$  i.e.  $\bar{x}_i \in \beta_i(\bar{p}, \bar{E})$ . Then one has  $\Phi_i(\bar{y}) = \alpha_i(\bar{p}, \bar{E}) \cap P_i(x)$  which is not empty. So the set  $\alpha_i(\bar{p}, \bar{E})$  is not empty, and since the set  $\{(x_i, p, E) \in X \times B \times G^J(R^{1+S}) \mid \alpha_i(p, E) \neq \emptyset\}$  is an open set of  $X \times B \times G^J(R^{1+S})$ , there exist an open neighborhood of  $\bar{y}$  called  $O'_{\bar{y}}$  such that if  $(x, p, E) \in O'_{\bar{y}}$  then  $\alpha_i(p, E) \neq \emptyset$ . Besides, since  $\alpha_i \cap P_i$  is lower semi-continuous as the intersection of an open graph correspondence and a lower semi continuous correspondence, there exists  $O''_{\bar{y}}$  an open neighbourhood of  $(\bar{x}, \bar{p}, \bar{q})$  such that if  $(x, p, q) \in O''_{\bar{y}}$  then  $U \cap (\alpha_i(p, q) \cap P_i(x)) \neq \emptyset$ . We then define  $O_{\bar{y}} = O'_{\bar{y}} \cap O''_{\bar{y}}$ .

Now, let  $y = (x, p, E) \in O_{\bar{y}}$ . If  $x_i \notin \beta_i(p, E)$  then  $\Phi_i(y) = \beta_i$ . Since  $\alpha_i(\bar{p}, \bar{E}) \cap P_i(x) \subset \beta_i(p, E)$  then  $\Phi_i(y) \cap U$  is not empty. If  $x_i \in \beta_i(p, E)$  then  $\Phi_i(y) = \alpha_i(\bar{p}, \bar{E}) \cap P_i(x)$  and so  $\Phi_i(y) \cap U$  is not empty which ends the proof  $\square$

### 3.2.2 The fixed-point argument

The existence proof relies on the following fixed-point theorem. If  $V$  is a Euclidean space, we let  $G^J(V)$  be the set consisting of all linear subspaces of  $V$  of dimension  $J$ .

**Theorem 3.2** *Let  $V$  be a Euclidean space, for  $i = 1, \dots, n$ , let  $X_i$  be a nonempty, compact, convex subset of some Euclidean space, and let  $X = \prod_{i=1}^n X_i \times G^J(V)$ .*

*For  $i = 1, \dots, n$ , let  $\Phi_i$  be a correspondence from  $X$  to  $X_i$ , which is lower semicontinuous and convex valued (possibly empty-valued), and for  $j = 1, \dots, J$ , let  $\psi_j : X \rightarrow V$  be a continuous mapping.*

*Then, there exist  $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n, \bar{E}) \in X_1 \times \dots \times X_n \times G^J(V)$  such that*

- (a) for every  $i = 1, \dots, n$ , [either  $\bar{x}_i \in \Phi_i(\bar{x})$  or  $\Phi_i(\bar{x}) = \emptyset$ ];*
- (b) for every  $j = 1, \dots, J$ ,  $\psi_j(\bar{x}) \in \bar{E}$ .*

The proof of Theorem 3.2 is given in the appendix as a consequence of a more standard result by Hirsch, Magill and Mas-colell (1987) or Husseini, Lasry and Magill (1986).

Let  $\lambda = (1, 1, \dots, 1) \in R_{++}^{1+S}$ . We let  $V = \lambda^\perp$  and we notice that  $V$  is a Euclidean space of dimension  $S$  and  $G^J(V) \subset G^J(R^{1+S})$ . We let  $n = m + 1$ , for  $i = 1, \dots, m$ ,  $X_i$  is taken to be the consumption set of the  $i$ th consumer which is convex, compact [from Assumption  $\hat{C}$ ] and nonempty [from the Survival Assumption  $SS$ ], and  $X_{m+1} = B$ , the closed unit ball of  $R^{K(1+S)}$ . From Lemma 3.1, the correspondences  $\Phi_i (i = 1, \dots, m + 1)$  and the mappings  $\psi_j (j = 1, \dots, J)$ , defined in Section 3.2.1, satisfy the assumptions of Theorem 3.2. So, there exists  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{E}) \in \prod_{i=1}^m X_i \times B \times G^J(V)$  such that, if we let  $\bar{x} = (\bar{x}_i)_{i=1}^m$  one has :

$$\bar{x}_i \in \beta_i(\bar{p}, \bar{E}) \text{ and } \alpha_i(\bar{p}, \bar{E}) \cap P_i(\bar{x}) = \emptyset, \text{ for every } i = 1, \dots, m; \text{ (FP1)}$$

$$p \cdot \sum_{i=1}^m (\bar{x}_i - e_i) \leq \bar{p} \cdot \sum_{i=1}^m (\bar{x}_i - e_i), \text{ for every } p \in B; \text{ (FP2)}$$

$$\psi_j(\bar{x}, \bar{p}, \bar{E}) \in \bar{E} \text{ for every } j = 1, \dots, J. \text{ (FP3)}$$

Let  $\bar{q} = (\bar{q}_1, \dots, \bar{q}_J)$  be defined by

$$\bar{q}_j = - \sum_{s=1}^S W_{s,j}^1(\bar{p}) \text{ for every } j = 1, \dots, J.$$

Then, in the next section, we shall show that  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$  is a pseudo-equilibrium of  $\mathcal{E}$ .

### 3.2.3 $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$ is a pseudo-equilibrium of $\mathcal{E}$

This is a consequence of the following Claims 3.1, 3.2 and 3.6.

**Claim 3.1**  $\text{Im } W(\bar{p}, \bar{q}) = \text{span } \{\psi_1(\bar{x}, \bar{p}, \bar{E}), \dots, \psi_J(\bar{x}, \bar{p}, \bar{E})\} \subset \bar{E}$ .

**Proof of Claim 3.1.** From our choice of  $\bar{q}$  we notice that:

$$\bar{\psi}_j := \psi_j(\bar{x}, \bar{p}, \bar{E}) = (\bar{q}_j, W_{1,j}^1(\bar{p}), \dots, W_{S,j}^1(\bar{p}))$$

is exactly the  $j$ -th column of the matrix  $W(\bar{p}, \bar{q})$ . Hence, the vector space spanned by the vectors  $\bar{\psi}_j$  is equal to  $\text{Im}W(\bar{p}, \bar{q})$ , the image of the matrix  $W(\bar{p}, \bar{q})$ . Consequently, the claim follows from the fixed-point Condition *FP3*.  $\square$

Condition *FP1* implies that  $\bar{x}_i \in \beta_i(\bar{p}, \bar{E})$ , for every  $i = 1, \dots, m$ , hence, from the definition of the sets  $\beta_i(\bar{p}, \bar{E})$ , there exists  $\bar{t}_i \in \bar{E}$  such that:

$$\bar{p} \square (\bar{x}_i - e_i) \leq \bar{t}_i + (1 - \|\bar{p}\|)1, \text{ for every } i = 2, \dots, m; \quad (2)$$

$$\bar{p} \cdot (\bar{x}_1 - e_1) \leq 1 - \|\bar{p}\|, \text{ for } i = 1. \quad (3)$$

Since  $\bar{t}_i \in \bar{E} \subset \lambda^\perp$  for every  $i = 1, \dots, m$ , taking the scalar product of each side of equations (1) with the vector  $\lambda \in R_{++}^{S+1}$ , we obtain:

$$\bar{p} \cdot (\bar{x}_i - e_i) \leq 1 - \|\bar{p}\|, \text{ for every } i = 2, \dots, m. \quad (4)$$

**Claim 3.2**  $\sum_{i=1}^m (\bar{x}_i - e_i) = 0$ .

**Proof of Claim 3.2.** If it is not true, then  $\sum_{i=1}^m (\bar{x}_i - e_i) \neq 0$  and it follows from Condition *FP2* that

$$\bar{p} = \frac{\sum_{i=1}^m (\bar{x}_i - e_i)}{\left\| \sum_{i=1}^m (\bar{x}_i - e_i) \right\|}.$$

Hence,  $\|\bar{p}\|=1$  and:

$$\bar{p} \cdot \sum_{i=1}^m (\bar{x}_i - e_i) = \left\| \sum_{i=1}^m (\bar{x}_i - e_i) \right\| > 0.$$

But summing up over  $i$  the above inequalities (3) and (4), and recalling that  $\|\bar{p}\|=1$ , we get

$$\bar{p} \cdot \sum_{i=1}^m (\bar{x}_i - e_i) \leq m(1 - \|\bar{p}\|) = 0,$$

which contradicts the above inequality.  $\square$

**Claim 3.3** *for every  $s = 0, 1, \dots, S$ ,  $\bar{p}(s) \neq 0$ .*

**Proof of Claim 3.3.** We first notice that  $\bar{p} \neq 0$ . Indeed, if  $\bar{p} = 0$ , then  $\alpha_i(\bar{p}, \bar{E}) = \beta_i(\bar{p}, \bar{E}) = X_i$ , for every  $i = 1, \dots, m$ . But, from claim 3.2,  $\bar{x} \in \mathcal{A}(\mathcal{E})$  and from the Non-Satiation Assumption there exists  $x_i \in P_i(\bar{x}) = P_i(\bar{x}) \cap \alpha_i(\bar{p}, \bar{E})$ . This contradicts Assertion *FP1*.

We now prove that for every  $s = 0, \dots, S$  we have  $\bar{p}(s) \neq 0$ . Indeed, suppose that for some  $s$ ,  $\bar{p}(s) = 0$ . From claim 3.2,  $\bar{x} \in \mathcal{A}(\mathcal{E})$ , and from the Non Satiation Assumption in state  $s$  for consumer 1 there exists  $x_1 \in P_1(\bar{x})$  such that  $x_1(s') = \bar{x}_1(s')$  for  $s' \neq s$ ; from Assertion *FP1*,  $\bar{x}_1 \in \beta_1(\bar{p}, \bar{E})$  and, recalling that  $\bar{p}(s) = 0$ , one deduces that  $x_1 \in \beta_1(\bar{p}, \bar{E})$ . But  $\alpha_1(\bar{p}, \bar{E}) \neq \emptyset$  from the Survival Assumption *SS* and the fact that  $\bar{p} \neq 0$ . We now let  $y_1 \in \alpha_1(\bar{p}, \bar{E})$  and we notice that  $[y_1, x_1[ \subset \alpha_1(\bar{p}, \bar{E})$ . But, from Assumption  $\hat{C}$ , the set  $P_1(\bar{x})$  is open in  $X_1$ , (recalling that  $x_1 \in P_1(\bar{x})$ ) hence  $[y_1, x_1[ \cap P_1(\bar{x}) \neq \emptyset$ . Consequently,  $P_1(\bar{x}) \cap \alpha_1(\bar{p}, \bar{E}) \neq \emptyset$ , which contradicts Assertion *FP1*.  $\square$

**Claim 3.4** *for every  $i = 1, \dots, m$   $\bar{x}_i \in \beta_i(\bar{p}, \bar{E})$  and  $\beta_i(\bar{p}) \cap P_i(\bar{x}) = \emptyset$ .*

**Proof of Claim 3.4.** From the fixed point condition *FP1*, one has  $\bar{x}_i \in \beta_i(\bar{p}, \bar{E})$ . Now, suppose that there exists  $i$  such that  $\beta_i(\bar{p}, \bar{E}) \cap P_i(\bar{x}) \neq \emptyset$ . Let  $x_i \in \beta_i(\bar{p}, \bar{E}) \cap P_i(\bar{x})$ . From the Survival Assumption *SS* and the fact that  $\bar{p}(s) \neq 0$  for every  $s = 0, \dots, S$ , [Claim 3.3] one deduces that  $\alpha_i(\bar{p}, \bar{E}) \neq \emptyset$  and we let  $y_i \in \alpha_i(\bar{p}, \bar{E})$ . We notice that  $[y_i, x_i[ \subset \alpha_i(\bar{p}, \bar{E})$ . But, from Assumption  $\hat{C}$ , the set  $P_i(\bar{x})$  is open, (recalling that  $x_i \in P_i(\bar{x})$ ) hence  $[y_i, x_i[ \cap P_i(\bar{x}) \neq \emptyset$ . Consequently,  $P_i(\bar{x}) \cap \alpha_i(\bar{p}, \bar{E}) \neq \emptyset$  which contradicts Assertion *FP1*.  $\square$

**Claim 3.5**  $\|\bar{p}\| = 1$ .

**Proof of Claim 3.5.** We first prove that the budget constraints of consumers  $i = 2, \dots, m$  are binded, that is:

$$\bar{p} \square (\bar{x}_i - e_i) = \bar{t}_i + (1 - \|\bar{p}\|)1, \text{ for every } i = 2, \dots, m.$$

Indeed, if it is not true, there exists  $i = 2, \dots, m$  and  $s$  such that  $\bar{p} \square (\bar{x}_i - e_i) \leq \bar{t}_i + (1 - \|\bar{p}\|)1$ , with a strict inequality for the  $s$ -th component. But  $\bar{x} \in \mathcal{A}(\mathcal{E})$  [claim 3.2] and from the Non-Satiation Assumption in state  $s$  (for consumer  $i$ ), there exists  $x_i \in X_i$  such that  $x_i \in P_i(\bar{x})$  and  $x_i(s') = \bar{x}_i(s')$  for every  $s' \neq s$ . Consequently, we can choose  $x \in [x_i, \bar{x}_i[$  close enough to  $\bar{x}_i$  so that  $\bar{x}_i \in \beta_i(\bar{p}, \bar{E})$ .

But from Assumption  $\hat{C}$ ,  $[x_i, \bar{x}_i] \subset P_i(\bar{x})$ . Consequently,  $\beta_i(\bar{p}, \bar{E}) \cap P_i(\bar{x}) \neq \emptyset$  which contradicts Claim 3.4.

In the same way, we prove that the budget constraint of consumer 1 is binded, that is

$$\bar{p} \cdot (\bar{x}_1 - e_1) = 1 - \|\bar{p}\|.$$

Now, multiplying the budget equalities of  $i = 2, \dots, m$  by  $\lambda \in R_{++}^{S+1}$  we obtain

$$\bar{p} \cdot (\bar{x}_i - e_i) = 1 - \|\bar{p}\|, \quad \text{for every } i = 2, \dots, m.$$

Summing up the above equations and using Claim 3.1 we get

$$0 = \bar{p} \cdot \sum_{i=1}^m (\bar{x}_i - e_i) = m(1 - \|\bar{p}\|).$$

Consequently,  $\|\bar{p}\| = 1$ .  $\square$

**Claim 3.6** *For every  $i = 1, \dots, m$ , the pseudo-equilibrium condition (i) is satisfied, that is,*

$$\bar{x}_i \in B_i(\bar{p}, \bar{E}) \text{ and } P_i(\bar{x}) \cap B_i(\bar{p}, \bar{E}) = \emptyset.$$

**Proof of Claim 3.6.** From Claim 3.5, for  $i = 2, \dots, m$ ,  $B_i(\bar{p}, \bar{E}) = \beta_i(\bar{p}, \bar{E})$ . Hence, claim 3.4 implies that Claim 3.6 is true for every consumer  $i = 2, \dots, m$ .

Let us now consider the first consumer. Noticing that  $B_1(\bar{p}, \bar{E}) \subset \beta_1(\bar{p}, \bar{E})$ , in view of claim 3.4, the proof will be complete if we show that  $\bar{x}_1 \in B_1(\bar{p}, \bar{E})$ , that is,

$$\bar{p} \square (\bar{x}_1 - e_1) \in \bar{E}.$$

But since  $\sum_{i=1}^m (\bar{x}_i - e_i) = 0$  [Claim 3.2] and since the budget constraint of every consumer is binded [cf. the proof of Claim 3.5] there exists  $\bar{t}_i \in \bar{E}$  such that:

$$\bar{p} \square (\bar{x}_1 - e_1) = - \sum_{i \neq 1} \bar{p} \square (\bar{x}_i - e_i) = - \sum_{i \neq 1} \bar{t}_i \in \bar{E}.$$

which finishes the proof  $\square$

Now, we do not suppose that  $r = J \leq S$ ; we will only suppose that  $S \geq r \geq r(W^1)$  and will prove Theorem 2.1.

First notice that by definition of  $r(W^1)$  there exist  $r(W^1)$  assets  $a_1(\cdot), \dots, a_{r(W^1)}(\cdot)$  such that for every  $p \in R^{K(1+S)}$ ,  $\text{Im}W^1(p) \subset \text{span}\{a_1(p), \dots, a_{r(W^1)}(p)\}$ .

Now, we can define a new economy with  $r$  assets by

$$\mathcal{E}' = ((X_i, P_i, e_i)_{i=1, \dots, m}, W'^1).$$

where the  $r(W^1)$  first columns of  $W'^1(p)$  are the  $a_i(p)$  ( $i = 1, \dots, r(W^1)$ ) and the other columns of  $W'^1(p)$  are all equal to  $a_1(p)$ .

We proved above that there exists a  $r$ -pseudo-equilibrium  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$  of  $\mathcal{E}'$ .

Now it is easy to see that  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}', \bar{E})$  is also a  $r$  pseudo-equilibrium of  $\mathcal{E}$ , where  $\bar{q}'_j = \sum_{i=1}^m W_{s,j}^1(\bar{p})$  for every  $j = 1, \dots, J$ .

□

## 4 APPENDIX

### 4.1 Proof of the fixed-point theorem

#### 4.1.1 Preliminaries

A correspondence  $\Phi$  from a set  $X$  to a set  $Y$  is a map from  $X$  to the set of all the subsets of  $Y$  and the graph of  $\Phi$ , denoted  $G(\Phi)$ , is defined by  $G(\Phi) = \{(x, y) \in X \times Y | y \in \Phi(x)\}$ . In the following we suppose that  $X$  and  $Y$  are metric spaces.<sup>3</sup> The correspondence  $\Phi$  is said to be upper semicontinuous (u.s.c.), [resp. lower semicontinuous (l.s.c.)] if the set  $\{x \in X | \Phi(x) \subset U\}$  [resp.  $\{x \in X | \Phi(x) \cap U \neq \emptyset\}$ ] is open in  $X$  for every open set  $U \subset Y$ .

Let  $V$  be a finite dimensional Euclidean space and let  $k$  be an integer such that  $0 \leq k \leq \dim(V)$ . We denote  $G^k(V)$  the set consisting of all the linear subspaces of  $V$  of dimension  $k$ , called the ( $k$ -)Grassmann manifold (in  $V$ ). Then it is known that  $G^k(V)$  is a smooth manifold of dimension  $k(\dim(V) - k)$ . A precise definition of the manifold structure on  $G^k(V)$  is given latter.

We first admit the following theorem (c.f Husseini, Lasry, Magill (1986) or Hirsch, Magill, Mas Colell (1987)).

**Theorem 4.1** *Let  $V$  be finite dimensional Euclidean space, let  $C$  be nonempty, compact, convex subset of some Euclidean space, and let  $X = C \times G^k(V)$ .*

*Let  $\phi : X \rightarrow C$  and for every  $j = 1, \dots, m$ ,  $\psi_j : X \rightarrow V$  be continuous mappings.*

<sup>2</sup>We let  $R_+ = \{x \in R | x \geq 0\}$ . If  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  belong to  $R^n$ , we denote  $(x|y) = \sum_{i=1}^n x_i y_i$ , the scalar product of  $R^n$ ,  $\|x\| = \sqrt{(x|x)}$ , the Euclidian norm; we denote  $B(x, r) = \{y \in R^n | \|x - y\| < r\}$ ,  $\bar{B}(x, r) = \{y \in R^n | \|x - y\| \leq r\}$  and  $S(x, r) = \{y \in R^n | \|x - y\| = r\}$ . If  $X \subset R^n$ ,  $Y \subset R^n$ , and  $x \in R^n$ , we let  $d(x, X) = \inf_{y \in X} \|x - y\|$ , we denote  $X \setminus Y = \{x \in X | x \notin Y\}$  the set-difference of the sets  $X$  and  $Y$ ,  $X + Y = \{x + y | x \in X, y \in Y\}$ , the sum of the sets  $X$  and  $Y$ ,  $B(X, r) = X + B(0, r)$ ,  $\bar{B}(X, r) = X + \bar{B}(0, r)$ ,  $\text{cl}X$  or  $\bar{X}$ , the closure of  $X$ ,  $\text{int}X$ , the interior of  $X$ ,  $\text{bd}X = \text{cl}X \setminus \text{int}X$ , the boundary of  $X$ ,  $X^\circ = \{y \in R^n | \forall x \in X, (y|x) \leq 0\}$ , the negative polar cone of  $X$ ,  $X^\perp = \{y \in R^n | \forall x \in X, (x|y) = 0\}$ , the orthogonal vector space to  $X$ ,  $\text{co } X$ , the convex hull of  $X$ .

<sup>3</sup>If  $\Phi$  and  $\Psi$  are two correspondences from  $X \subset R^n$  to  $R^m$ , the correspondences  $\Phi \cap \Psi$ ,  $\text{co } \Phi$ , are defined, respectively, by  $(\Phi \cap \Psi)(x) = \Phi(x) \cap \Psi(x)$ ,  $(\text{co } \Phi)(x) = \text{co } \Phi(x)$ . A map  $\varphi : X \rightarrow R^m$  is called a selection of  $\Phi$ , if  $\varphi(x) \in \Phi(x)$  for all  $x \in X$ . If  $A$  is a subset of  $X$ , we denote  $\Phi(A) = \cup_{x \in A} \Phi(x)$  and we define the restriction of  $\Phi$  to  $A$ , denoted  $\Phi|_A$ , to be the correspondence from  $A$  to  $R^m$  defined by  $\Phi|_A(x) = \Phi(x)$  if  $x \in A$ .

Then, there exist  $\bar{x} = (\bar{x}_1, \bar{E}) \in X$  such that

- (a)  $\bar{x}_1 \in \phi(\bar{x})$ ;
- (b) for every  $j = 1, \dots, m$ ,  $\psi_j(\bar{x}) \in \bar{E}$ .

#### 4.1.2 Proof of the fixed-point-like theorem

We first prove the following extension of Theorem 4.1:

**Theorem 4.2** *Let  $V$  be finite dimensional Euclidean space, let  $C$  be nonempty, compact, convex subset of some Euclidean space, and let  $X = C \times G^k(V)$ .*

*Let  $\Phi$  be a correspondence from  $X$  to  $C$ , which is upper semicontinuous and nonempty, convex, compact valued, and for  $j = 1, \dots, m$  let  $\psi_j : X \rightarrow V$  be a continuous mapping.*

Then, there exist  $\bar{x} = (\bar{x}_1, \bar{E}) \in X$  such that

- (a)  $\bar{x}_1 \in \Phi(\bar{x})$
- (b) for every  $j = 1, \dots, m$   $\psi_j(\bar{x}) \in \bar{E}$ .

**Proof of Theorem 4.2.** From Cellina (1969), for every  $\epsilon > 0$  there exists a continuous mapping  $\varphi^\epsilon : X \rightarrow C$  such that for every  $x \in X$  we have  $Gr(\varphi^\epsilon) \subset Gr(\Phi) + \epsilon B(0, 1)$ . We apply Theorem 4.1 to the mappings  $\varphi^\epsilon$  and  $\psi_j$  ( $j = 1, \dots, m$ ). Then we obtain  $(\bar{x}_1^\epsilon, \bar{E}^\epsilon) \in X$  that satisfies:

- (a)  $\bar{x}_1^\epsilon \in \varphi^\epsilon(\bar{x}_1^\epsilon, \bar{E}^\epsilon)$
- (b) for every  $j = 1, \dots, m$   $\psi_j(\bar{x}_1^\epsilon, \bar{E}^\epsilon) \in \bar{E}^\epsilon$ .

Since  $X$  is compact we can suppose without any loss of generality that the sequence  $(\bar{x}_1^\epsilon, \bar{E}^\epsilon)$  converges, when  $\epsilon$  converges to 0, to an element  $(\bar{x}_1, \bar{E})$  that satisfies the conclusion of Theorem 5.2.  $\square$

**Proof of Theorem 3.2.** For every  $i = 1, \dots, n$ , let  $U_i = \{x \in X_1 \times \dots \times X_n \mid \text{there exists } E \in G^J(V), \varphi_i(x, E) \neq \emptyset\}$ , then  $\Phi \mid U_i \times G^J(V) : U_i \times G^J(V) \rightarrow X_i$  is a convex valued correspondence having an open graph. Let  $f_i : U_i \times G^J(V) \rightarrow X_i$  be a continuous function such that  $f_i(x, E) \in \Phi_i(x, E)$  for every  $(x, E) \in U_i \times G^J(V)$  [see Michael]. For  $i = 1, \dots, n$ , define correspondences  $\Phi_i : X \rightarrow X_i$  by  $\Phi_i(x, E) = f_i(x, E)$  if  $x \in U_i$  and  $\varphi_i(x, E) = X_i$  otherwise. For every  $i$ ,  $\Phi_i$  is non-empty, convex valued and upper-semicontinuous. Then we can apply Theorem 4.1 to  $\Phi : X \rightarrow \prod X_i$  define by  $\Phi(x) = \prod_{i=1}^n \Phi_i(x)$  and to the mappings  $\psi_j$ .

Then there exists  $(\bar{x}, \bar{E}) = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n, \bar{E}) \in X$  such that

- (a)  $\bar{x} \in \Phi(\bar{x})$ ;
- (b) for every  $j = 1, \dots, J$ ,  $\psi_j(\bar{x}) \in \bar{E}$ .

By construction,  $(\bar{x}, \bar{E})$  satisfies the conclusion of the Theorem 3.2.  $\square$

### 4.1.3 Link with the usual concept of pseudo-equilibrium

In the literature, we often meet the following definition of pseudo-equilibrium, which is slightly different from the one used here.

**Definition 4.1** *A pseudo-equilibrium' of the economy  $\mathcal{E}$  is an element  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{F})$  of  $\prod_{i=1}^m X_i \times R^{K(1+S)} \times G^J(R^S)$  such that, if we let  $\bar{x} = ((\bar{x}_i)_{i=1}^m)$ , one has:*

(i) *for every  $i = 1, \dots, m$ ,  $\bar{x}_i$  is a "maximal" element of  $P_i$  in the budget set*

$$B'_i(\bar{p}, \bar{F}) := \left\{ x_i \in X_i \mid \bar{p}^1 \square (x_i^1 - e_i^1) \in \bar{F}, \bar{p} \cdot (x_i - e_i) = 0 \right\}^4,$$

*in the sense that:*

$$\bar{x}_i \in B'_i(\bar{p}, \bar{F}) \text{ and } P_i(\bar{x}) \cap B'_i(\bar{p}, \bar{F}) = \emptyset;$$

(ii)  $\sum_{i=1}^m \bar{x}_i = \sum_{i=1}^m e_i$ ;

(iii)  $\text{Im } W^1(\bar{p}) := \{ W^1(\bar{p})z \mid z \in R^J \} \subset \bar{F}$ .

The link between the two notions of Pseudo-equilibria is given in the next proposition.

**Proposition 4.1** *Under the Non-satiation assumption, if  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$  is a pseudo-equilibrium of  $\mathcal{E}$ , with  $\bar{\lambda} \in \bar{E}^\perp$ , then  $((\bar{x}_i)_{i=1}^m, \bar{\lambda} \square \bar{p}, \bar{\lambda}^1 \square \bar{E}^1)^5$  is a pseudo-equilibrium' of  $\mathcal{E}$ .*

*Conversely, let  $\bar{\lambda} \in R^{S+1}$  and  $\bar{\lambda}' = (1/\bar{\lambda}(0), \dots, 1/\bar{\lambda}(S))$ ; if we assume the monotonicity assumption and if  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{F})$  is a pseudo-equilibrium' of  $\mathcal{E}$ , then  $((\bar{x}_i)_{i=1}^m, \bar{\lambda}' \square \bar{p}, \bar{q}, \bar{E})$  is a pseudo-equilibrium of  $\mathcal{E}$ , where  $\bar{E} = \{(x(0), x(1)/\bar{\lambda}(1), \dots, x(S)/\bar{\lambda}(S)) \mid (x_1, \dots, x_S) \in \bar{F} \text{ and } x_0 = -1/\bar{\lambda}(0) \sum_{i=1}^S x_i\}$ . Furthermore, we have  $\bar{\lambda} \in \bar{E}^\perp$ .*

**Proof of Proposition 4.1** Let  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{q}, \bar{E})$  be a pseudo-equilibrium of  $\mathcal{E}$ . Then we have the condition (ii) of pseudo-equilibrium':  $\sum_{i=1}^m \bar{x}_i = \sum_{i=1}^m e_i$ . Secondly, the condition  $\text{Im}(W(\bar{p}, \bar{q})) \subset \bar{E}$ , where  $W(\bar{p}, \bar{q}) = \begin{pmatrix} -\bar{q} \\ W^1(\bar{p}) \end{pmatrix}$ , implies that  $\text{Im}(W^1(\bar{p})) \subset \bar{E}^1$  where  $\bar{E}^1 := \{(x_1, \dots, x_S), (x_0, \dots, x_S) \in \bar{E}\}$  is a linear space of dimension  $J$ . So,  $\text{Im}(W^1(\bar{\lambda}' \square \bar{p})) \subset \bar{\lambda}^1 \square \bar{E}^1$  where  $\bar{\lambda}^1 \square \bar{E}^1 = \{(\bar{\lambda}(1)x_1, \dots, \bar{\lambda}(S)x_S), (x_1, \dots, x_S) \in \bar{F}\}$  is also a linear space of dimension  $J$ , and we obtain the condition (iii) of pseudo-equilibrium'. Now, under Non-satiation assumption we have for every  $i = 1, \dots, m$ :  $\bar{p} \square (\bar{x}_i - e_i) \in \bar{E}$ . Since  $\bar{\lambda} \in \bar{E}^\perp$  we obtain  $(\bar{\lambda} \square \bar{p}) \cdot (x_i - e_i) = 0$  and we obviously obtain  $(\bar{\lambda} \square \bar{p})^1 \square (x_i^1 - e_i^1) \in \bar{\lambda}^1 \square \bar{E}^1$ . So we have  $\bar{x}_i \in B'_i(\bar{\lambda} \square \bar{p}, \bar{\lambda}^1 \square \bar{E}^1)$ .

<sup>4</sup>We define  $\bar{p}^1 \square (x_i^1 - e_i^1) = (\bar{p}(1) \cdot (x_i^1(1) - e_i^1(1)), \dots, \bar{p}(S) \cdot (x_i^1(S) - e_i^1(S)))$ .

<sup>5</sup>We define  $\bar{\lambda}^1 \square \bar{E}^1 = \{(\lambda(1)x(1), \dots, \lambda(S)x(S)), (x(0), \dots, x(S)) \in \bar{E}\}$ .

Finally, we note that  $B'_i(\bar{\lambda} \square \bar{p}, \bar{\lambda} \square \bar{E}^1) \subset B_i(\bar{p}, \bar{E})$  which implies that the condition (i) of pseudo-equilibrium is satisfied:  $P_i(\bar{x}) \cap B'_i(\bar{\lambda} \square \bar{p}, \bar{\lambda}^1 \square \bar{E}^1) = \emptyset$ . So,  $((\bar{x}_i)_{i=1}^m, \bar{\lambda} \square \bar{p}, \bar{\lambda}^1 \square \bar{E}^1)$  is a pseudo-equilibrium' of  $\mathcal{E}$ .

Conversely, suppose that  $((\bar{x}_i)_{i=1}^m, \bar{p}, \bar{F})$  is a pseudo-equilibrium'. We first have the condition (ii) of pseudo-equilibrium:  $\sum_{i=1}^m \bar{x}_i = \sum_{i=1}^m e_i$ . Now,  $\bar{q}$  is perfectly defined by the equality  $\bar{\lambda}.W((1/\bar{\lambda}) \square \bar{p}, \bar{q}) = 0$ . We now define  $\bar{E} = \{(x_0, x_1/\bar{\lambda}(1), \dots, x_S/\bar{\lambda}(S)) \mid (x_1, \dots, x_S) \in \bar{F} \text{ and } x(0) = -1/\bar{\lambda}(0) \sum_{i=1}^S x_i\}$  It is easy to prove that  $\text{Im}W((\bar{\lambda}') \square \bar{p}, \bar{q}) \subset \bar{E}$  and that  $\bar{x}_i \in B_i((\bar{\lambda}') \square \bar{p}, \bar{E})$ .

Finally, if  $P_i(\bar{x}) \cap B_i(\bar{\lambda}' \square \bar{p}, \bar{E}) \neq \emptyset$ , then the monotonicity assumption implies that  $P_i(\bar{x}) \cap B'_i(\bar{p}, \bar{F}) \neq \emptyset$ , a contradiction.  $\square$ .

#### 4.1.4 Annex 2 : $\mathcal{E}$ and $\mathcal{E}$ have the same equilibria

## References

- [1] BERGSTROM, T.C. (1976): "How to discard "free disposability" - at no cost", *Journal of Mathematical Economics* 3, 131-134.
- [2] CASS, D. (1984): "Competitive equilibrium with incomplete financial markets," CARESS Working paper 84-09, University of Pennsylvania.
- [3] CELLINA, A. (1969): "Approximation of set-Valued and fixed point Theorem" *Annale di Mathematica Pura ed Applicata*, 4, 17-24.
- [4] DEBREU, G. (1959): *Theory of Value*, Wiley, New-York.
- [5] DUFFIE, D. (1992): *Asset Pricing Theory*, Princeton University Press, Princeton.
- [6] DUFFIE, D. AND W. SHAFFER (1985): *Equilibrium in incomplete markets I: basic model of generic existence*, *Journal of Mathematical Economics*, 14:285-300.
- [7] FLORENZANO, M. (1981): *L'équilibre économique général transitif et in-transitif : problèmes d'existence*, Monographies du séminaire d'économétrie, Editions du C.N.R.S., Paris.
- [8] FLORENZANO, M., P. GOURDEL AND V. MARAKULIN (1996): "Limits of Radner equilibria and equilibrium existence in incomplete markets, Working paper, Paris.
- [9] GALE, D. (1960): "The theory of Linear Economic Models", Mc Graw Hill Book Company.
- [10] GALE, D. AND A. MAS-COLELL (1975): "An equilibrium existence theorem for a general model without ordered preferences", *Journal of Mathematical Economics* 2, 9-15.

- [11] GALE, D. AND A. MAS-COLELL (1979): "Corrections to an equilibrium existence theorem for a general model without ordered preferences", *Journal of Mathematical Economics* 6, 297-298.
- [12] HARRISON, M. AND D. KREPS (1979): "Martingale and Arbitrage in Multiperiod Security Markets", *Journal of Economic Theory* 20: 381-408.
- [13] GEANAKOPOLOS, J. (1990): "An introduction to General Equilibrium with incomplete asset markets", *Journal of Mathematical Economics* 19, 1-38.
- [14] HART, O. (1975): "On the optimality of equilibrium when the market structure is incomplete", *Journal of Economic Theory* 11, 418-443.
- [15] HIRSCH, M., M. MAGILL AND A. MAS-COLELL (1987): "A geometric approach to a class of equilibrium existence theorems.", *Journal of Mathematical Economics* 19, 95-106.
- [16] HUSSEINI S.Y., J.M LASRY AND M.MAGILL (1986): "Existence of equilibrium with incomplete markets", *Journal of Mathematical Economics* 19, 39-67.
- [17] KEIDING, H. (1992): "Existence of pseudo-equilibria in economies with incomplete markets", Working paper, Copenhagen.
- [18] MAGILL, M., AND SHAFER, W. (1991): "Incomplete Markets," in *Handbook of Mathematical Economics*, Volume 4, edited by W. Hildenbrand and H. Sonnenschein. 19:167-194.
- [19] MARAKULIN, V. (1994): "On the notion of generalized equilibrium in incomplete markets and its existence theorem", Working paper, Novosibirsk.
- [20] OIKO NOMIA (1996): "Introduction to financial markets".
- [21] OIKO NOMIA (1996): "Financial equilibrium : bounded portfolios and the limiting case".
- [22] OIKO NOMIA (1997): "Existence of Walras equilibria : simultaneous optimization".
- [23] POLEMARCHAKIS, H (1988): "Portfolio choice, exchange rates and indeterminacy".
- [24] RADNER (1972): " Existence of equilibrium of plans, prices and price expectations in a sequence of markets", *Econometrica* 40, no. 2, 289-303.